Furstenberg's $\times 2, \times 3$ type Conjectures
Furstenberg's Slicing Conjecture and our new results
A few words about the proofs

Furstenberg-Marstrand slicing Theorems for $(\times m, \times n)$ invariant sets

Amir Algom

The Pennsylvania State University

Joint work with Meng Wu, University of Oulu, Finland
Furstenberg’s $\times 2, \times 3$ type Conjectures

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Furstenberg’s Slicing Conjecture and our new results

For an integer $n \geq 2$ we define, for $x \in [0,1)$

$$T_n(x) = n \cdot x \mod 1$$

In base $n$ this is

$$T_n(0.x_1x_2x_3...) = 0.x_2x_3x_4...$$

A set $X \subseteq [0,1)$ is $T_n$ invariant if it is closed and $T_n(X) = X$.

For example, $C = \{ \sum_{k=1}^{\infty} x_k 3^k, \text{where } x_k \in \{0,2\} \}$ is $T_3$ invariant.

A measure $\mu$ is $T_n$ invariant if $T_n\mu = \mu$.

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Let $A$ be a $T_m$ invariant set, and let $B$ be a $T_n$ invariant set.
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Let $A$ be a $T_m$ invariant set, and let $B$ be a $T_n$ invariant set. If $m \not\sim n$ then $A$ and $B$ should have no common structure. Similarly, if $\mu$ is $T_m$ invariant, and $\nu$ is $T_n$ invariant, then $\mu$ and $\nu$ should share no common structure.
The $\times 2, \times 3$ Conjecture

Theorem (Furstenberg, 1967)
Let $X \subseteq T := \mathbb{R}/\mathbb{Z}$ be a closed set that is invariant under both $T_m$ and $T_n$. If $m \not\sim n$ then either $X$ is finite or $X = T$.

Conjecture (Furstenberg, 1967)
If $\mu$ is a Borel probability measure on the circle invariant under $T_m$ and $T_n$. If $m \not\sim n$ then it is a convex combination of the Lebesgue measure and a purely atomic measure.

In the early 1990's, Rudolph and Johnson proved the Conjecture holds true in the positive entropy case. The zero entropy case remains open.
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1. Let $r > 0$. For a bounded set $A \subseteq \mathbb{R}^d$ let $N_r(A)$ denote the minimal amount of sets of diameter $r$ required to cover $A$.

2. The box dimension of $A$ is defined as $dim_B A = \lim_{r \to 0} \frac{\log N_r(A)}{-\log r}$.

3. The Hausdorff dimension of a set $A$ is denoted by $dim_H A$, and $dim_P A$ will denote the packing dimension of $A$.

4. In general, $dim_H A \leq dim_P A \leq dim_B A$.

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Let $X$ be a closed $T_m$ invariant set. Then
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Furstenberg’s Conjecture about sum sets

Let $X, Y$ be closed sets that are invariant under $T_m$ and $T_n$, respectively. If $m \not\sim n$ then
\[
\dim H(X + Y) \leq \min\{\dim H X + \dim H Y, 1\}. \tag{1}
\]

For any $X, Y$ the RHS of (1) is an upper bound. $X + Y$ is the image of $X \times Y$ under the projection $(x, y) \mapsto x + y$.

By Marstrand's projection Theorem (1954), for all Borel sets $A, B \subseteq \mathbb{R}$:

for Lebesgue almost every $u$, \[
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If $X$ and $Y$ have no common structure, then (1) "should" hold (but not vice versa!).
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On the resolution of the sum set Conjecture

An $n$-adic Cantor set is a set of the form, for $D \subseteq \{0, \ldots, n-1\}$

$$\infty \sum_{k=1}^{n} x_k n_k : x_k \in D$$

In 2009 Shmerkin and Peres proved the Conjecture when $X$ and $Y$ are $m$-adic and $n$-adic Cantor sets, respectively. The proof relied on the work of Moreira, about sumsets of non-linear Cantor sets (1998).

Theorem (Hochman-Shmerkin, 2012)
The sum-set Conjecture holds true. In fact, for all $u \neq 0$

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Furstenberg's Slicing Conjecture (Furstenberg, 1969)

Let $X, Y$ be closed sets that are invariant under $T^m$ and $T^n$, respectively. If $m \not\sim n$ then for any line $\ell$ not parallel to the major axes $\dim H_\ell \cap (X \times Y) \leq \max\{\dim H_X + \dim H_Y - 1, 0\}$.

If the sumset $X + Y$ is "large" then "many" fibers $\ell_z = \{(x, y) : x \in X, y \in Y, x + y = z\}$ should be small, and vice versa. So (Sumset Conjecture) \(\iff\) (Slicing Conjecture).

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Sum-set versus slice
Furstenberg’s Slicing Conjecture

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Furstenberg's Slicing Conjecture

By Marstrand's slicing Theorem (1954) for all $X \subseteq \mathbb{R}^2$ and any line with fixed slope $u$, for almost every intercept $t$

$$\dim H X \cap \ell_{u,t} \leq \max \{ \dim H X - 1, 0 \}$$

This fails for any value smaller than the RHS of (3).

It is well known that for sets $X$ and $Y$ as in the Conjecture

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So, what Furstenberg Conjectured is that for $X = X_1 \times X_2$ as in the Conjecture, Marstrand's Theorem holds for all lines not parallel to the major axes.

In particular, a slice that violates (3) can be seen as some shared structure between $X$ and $Y$.  

Amir Algom

Furstenberg-Marstrand slicing Theorems for $(\times m, \times n)$ invariant sets
Furstenberg’s Slicing Conjecture

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Furstenberg's Slicing Conjecture and our new results

Some results towards slicing Conjecture

Furstenberg proved: fix $\ell_0$ such that $\dim H_\ell \cap (X \times Y) \geq \alpha$, then for Lebesgue a.e. $u$ there exists a $t$ such that $\dim H_{\ell u,t} \cap (X \times Y) \geq \alpha$.

In 1996 Wolff proved that for every line $\ell$ not parallel to the axes $\dim H_\ell \cap (X \times Y) \leq \max\{\dim H_X + \dim H_Y - \frac{1}{2}, 0\}$.

In 2014, Feng Huang and Rao proved that for every such $X,Y$ there is some (non-effective) $\delta = \delta(X,Y) > 0$ such that $\dim H_\ell \cap (X \times Y) \leq \min\{\dim X, \dim Y\} - \delta(X,Y)$.
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Let $X, Y$ be closed sets that are invariant under $T^m$ and $T^n$, respectively.
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Amir Algom
Bedford-McMullen carpets

Let $m, n \geq 2$ be integers. Let $\emptyset \neq D \subseteq \{0, ..., m-1\} \times \{0, ..., n-1\}$ and define $F = \{ (\sum_{k=1}^{\infty} x_k m^k, \sum_{k=1}^{\infty} y_k n^k) : (x_k, y_k) \in D \}$. $F$ is a (Bedford-McMullen) carpet with defining exponents $m, n$ and allowed digit set $D$. In 1984 Bedford and McMullen calculated their dimension. Warning: It is not true in general that $\dim_H F = \dim_B F$. But $\dim_B F = \dim_P F$ holds true.
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The Hironaka Curve

\[ D = \{(0,1,1),(1,1,0)\} \]

\[ p_0 = 700,000 \]

\[ p_1 = 713 \]

\[ m = 2 \]

\[ n = 3 \]

\[ \text{dim}_m F = \log_2(1 + 1 + \frac{1}{2}) = 1 + \log_2(2) = 1 + \log_3(2) = 1.585 \]
Past work on Slicing Theorems for carpets

Let $\Pi_2: \mathbb{R}^2 \to \mathbb{R}$ denote the projection $\Pi_2(x,y) = y$. For every $j \in \Pi_2(D)$, let $D_j = \{0 \leq i \leq m - 1 : (i,j) \in D\}$.

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Let $F$ be a carpet with exponents $m,n$ and digits $D$. If $m \not\sim n$ and for all $i,j$ we have $|D_j| = |D_i|$, then for any non-principal line $\ell$ $\dim B(\ell \cap F) \leq \max\{\dim H F - 1, 0\}$.

The proof also gives a bound when $|D_j| \neq |D_i|$.

The condition $|D_j| = |D_i|$ holds when $F$ is a product set of $m$-adic and $n$-adic Cantor sets. Thus, this generalises the result of Wu, that proved the slicing Conjecture.
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A few words about the proofs

Theorem (Kenyon-Peres, 1996)

\[ \lim_{k \to \infty} \dim H_{F_k} = \dim H_X. \]

By the slicing Theorem for carpets, which applies since

\[ m_k \not\sim n_k \]

for all \( k \),

\[ \dim H_X \cap \ell_0 \leq \dim H_{F_k} \cap \ell_0 \leq \max\{\dim H_{F_k} - 1, 0\} \]

and by Theorem of Kenyon and Peres

\[ \max\{\dim H_{F_k} - 1, 0\} \to \max\{\dim H_X - 1, 0\} \]

which concludes the proof.
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On the proof of the slicing Theorem for carpets

Fix a carpet $F$ with digit set $D$ and exponents $m \not\sim n$.
We always assume that $m > n$ so
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Let $\ell_0 \subseteq \mathbb{R}^2$ be a line with slope $m u_0$ where $u_0 \in [0, 1)$.
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For every $u \in T := \mathbb{R} / \mathbb{Z}$, we define a map $\Phi_u : [0, 1]^2 \to [0, 1]^2$ by

$$\Phi_u(x, y) = \begin{cases} 
(T m(x), T n(y)) & \text{if } u \in [1 - \theta, 1) \\
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So $\Phi_k u(x, y) \approx (T \lfloor k \cdot \theta \rfloor m(x), T k n(y))$.

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Furstenberg-Marstrand slicing Theorems for $(\times m, \times n)$ invariant sets
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where \( D_p(w) \) is the unique cell of the partition

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that contains $w$, and the measure $\mu^{D_m(x) \times D_n(y)}$ is the pushfoward via $T_m \times T_n$ of the conditional measure of $\mu$ on $D_m(x) \times D_n(y)$ ($\mu^{[0,1] \times D_n(y)}$ is defined similarly).
Furstenberg’s Slicing Conjecture and our new results

A few words about the proofs

Let \( \mu_0 \) be a measure on \( \ell_0 \cap F \) such that

\[
\dim \mu_0 := \inf \{ \dim H_A : \mu(A) > 0 \} \approx \dim \ell_0 \cap F
\]

and choose \((x_0, y_0) \sim \mu_0\).

Theorem (Furstenberg, 1969) - CP distributions

There exists a sequence \( N_j \) such that:

\[
\frac{N_j - 1}{\sum k=0} \delta M_k(\mu_0, (x_0, y_0), u_0) \to Q
\]

with

\[
\int (\dim \mu) dQ(\mu, z, u) \approx \dim \mu_0.
\]

Moreover, for \( Q \) almost every \((\mu, z, u)\) the measure \( \mu \) is supported on a line with slope \( m_u \) passing thorough the point \( z \).

Furthermore, \( u \) is distributed according to Lebesgue.
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**Theorem (Furstenberg, 1969) - CP distributions**

There exists a sequence \( N_j \) such that:

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A few words about the proofs

Crucial inequality

Consider the measures

\[ \sum_{k=0}^{\delta_T} \delta_{T_k}(y_0) \rightarrow \nu, \]

and

\[ \sum_{k=1}^{\delta_T} \delta_{T_k}(y_0) \rightarrow \rho.\]

Recall that

\[ D_j = \{ 0 \leq i \leq m-1 : (i,j) \in D \}. \]

Theorem (A. - Wu)

We have

\[ \int (\dim \mu) dQ(\mu) + 1 - o(1) \leq \sum_{j=0}^{n-1} \nu([j]_n, [j+1]_n) \log |D_j| \log m + h(\rho, T_n) \log n. \]

where the $o(1)$ error may be made arbitrarily small.

The RHS of (4) "can be shown" to be bounded above by

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Motivation behind crucial inequality
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Consider $M^k(\mu_0, (x_0, y_0), u_0)$. Then a product structure begins to emerge in the support of the measure component in a \textit{quantitative} manner. Namely, assuming $u_0 = 0$, for every $k \in \mathbb{N}$, the set

$$ \Pi_1$$

of the support of the measure component of $M^k(\mu_0, z_0, u_0)$ is contained in the set

$$ \left\{ \sum_{i=1}^{\infty} \frac{b_i}{m^i} : b_i \in \{0, \ldots, m - 1\} \text{ and for } 1 \leq i \leq n_k, b_i \in D_{T_n^{i+[k\theta]}}(y_0) \right\}$$

and $n_k \to \infty$ as $k \to \infty$. 
Motivation behind crucial inequality

For $y = \sum_{k=1}^{\infty} y_k n_k$, $y_k \in \Pi_2(D)$ define $A(y) = \{\sum_{i=1}^{\infty} b_i m_i: b_i \in D, y_i\}$. Morally, for $Q$-a.e. $(\mu, z, u)$:

1. $\mu$ is supported on a line with slope $m_u$ passing through the point $z$.
2. $\int (\dim \mu) dQ(\mu) \approx \dim \mu_0 \approx \dim \ell_0 \cap F$.
3. $\mu$ is supported on a product set $A(y) \times \Pi_2(F)$ with $y \sim \nu$ and $\Pi_2 \mu$ can be related to $\rho$. 
Motivation behind crucial inequality

For \( y = \sum_{k=1}^{\infty} \frac{y_k}{n_k}, y_k \in \Pi_2(D) \) define

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