Oseledets Spectrum for Perron-Frobenius Cocycles

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April 16th 2020
The set up: \( \sigma : \Omega \rightarrow \Omega \) is an (invertible) ergodic measure-preserving transformation.

\( A : \Omega \rightarrow M_d(\mathbb{R}) \) is a map into the \( d \times d \) matrices.

A *cocycle* is defined by \( A(n, \omega) = A^{(n)}(\omega) = A(\sigma^{n-1}\omega) \cdots A(\omega) \).

If \( A \) maps into invertible matrices and \( \sigma \) is invertible, may define also for negative \( n \).

\[
A(n + m, \omega) = A(m, \sigma^n \omega)A(n, \omega).
\]

What can be said about the typical behaviour of \( A^{(n)}(\omega) \)?
Theorem (Oseledets)

Let $\sigma : \Omega \to \Omega$ be measurable and invertible; $\mathbb{P}$ an ergodic invariant measure. $A : \Omega \to M_{d \times d}(\mathbb{R})$ is measurable.

Then there exist:

- $\lambda_1 > \lambda_2 > \ldots > \lambda_k \geq -\infty$; $m_1 + \ldots + m_k = d$
- $\mathbb{R}^d = V_1(\omega) \oplus V_2(\omega) \oplus \ldots \oplus V_k(\omega)$ (equivariant: $A_\omega(V_i(\omega)) \subset V_i(\sigma(\omega))$); $\text{dim } V_i = m_i$.
- $x \in V_i(\omega) \setminus \{0\}$ implies $\frac{1}{n} \log \|A_\omega^{(n)} x\| \to \lambda_i$. 

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Expansion rate $\lambda_1$
Expansion rate $\lambda_2$
Ruelle, Mañé, Thieullen, Lian and Lu, Blumenthal, Froyland, González-Tokman, Lloyd and Q proved versions of the MET where the matrices are replaced by operators on a Banach space. The operators are required to satisfy some *compactness* or *quasi-compactness* properties.

\[ X = V_1(\omega) \oplus V_2(\omega) \oplus \ldots \oplus R(\omega). \]

(Compare \( R(\omega) \) to the essential spectrum of an operator on a Banach space).
Quasi-compactness by picture
We want to apply these operator-valued METs to Perron-Frobenius operators. The operator $\mathcal{L}$ sends densities to densities: if $X$ is an absolutely continuous random variable with density $f$, then $T(X)$ is absolutely continuous with density $\mathcal{L}f$.

Perron-Frobenius operators for expanding maps tend to increase regularity, leading to the required quasi-compactness.
For Perron-Frobenius operators satisfying a Lasota-Yorke inequality, (weak norm $= |\cdot|$; strong norm $= \|\cdot\|$).

Define $\|\mathcal{L}^\epsilon - \mathcal{L}\| = \sup_{\|f\|=1} |(\mathcal{L}^\epsilon - \mathcal{L})f|$.

If $\|\mathcal{L}^\epsilon - \mathcal{L}\|$ is small then the peripheral spectra of $\mathcal{L}^\epsilon$ and $\mathcal{L}$ are close (acting on $(X, \|\cdot\|)$).
Is there a random version of Keller-Liverani?
Bochi, in his thesis, proved:

**Theorem (Bochi)**

Let $\Omega$ be a compact Hausdorff space, equipped with a Borel probability measure $\mathbb{P}$. Let $\sigma : (\Omega, \mathbb{P}) \to (\Omega, \mathbb{P})$ be an ergodic invertible measure-preserving transformation. There is a dense $G_\delta$ subset, $\mathcal{R}$ of $C(X, SL_2(\mathbb{R}))$ such that for $A \in \mathcal{R}$, either:

- The cocycle with generator $A$ is hyperbolic; or
- The cocycle has both Lyapunov exponents 0.

That is, away from hyperbolic cocycles, the Lyapunov exponents are discontinuous.

*Carefully chosen perturbations may collapse the Lyapunov spectrum.*
Theorem (Ledrappier and Young - simplified version)

Let $\Omega$ be a compact metric space; $\sigma$ a homeomorphism with an ergodic invariant measure $\mathbb{P}$. Let $A: \Omega \rightarrow \text{GL}_d(\mathbb{R})$ be continuous. Let the Lyapunov exponents be $\lambda_1 > \ldots > \lambda_d$ be the Lyapunov exponents of the cocycle $A^{(n)}(\omega)$.

Suppose $(P_n)$ is an i.i.d. sequence of matrices with absolutely continuous distribution.

Then let $A^\epsilon(\omega) = A(\omega) + \epsilon P$ and let $\lambda_1^\epsilon > \ldots > \lambda_d^\epsilon$ be the exponents of the perturbed cocycle.

Then $\lambda_i^\epsilon \rightarrow \lambda_i$ as $\epsilon \rightarrow 0$.

Lyapunov exponents are stable under noise-like perturbations.
Theorem (Froyland, González-Tokman, Q)

Let \((\Omega, \mathbb{P})\) be a measure space; \(\sigma\) an ergodic invertible measure-preserving transformation. Let \(A: \Omega \rightarrow M_d(\mathbb{R})\) be measurable. Let the Lyapunov exponents be \(\lambda_1 > \ldots > \lambda_d\) be the Lyapunov exponents of the cocycle \(A^{(n)}(\omega)\).

Suppose \((P_n)\) is an i.i.d. sequence of matrices with absolutely continuous distribution.

Then let \(A^\varepsilon(\omega) = A(\omega) + \varepsilon P\) and let \(\lambda^\varepsilon_1 > \ldots > \lambda^\varepsilon_d\) be the exponents of the perturbed cocycle.

Then \(\lambda^\varepsilon_i \rightarrow \lambda_i\) as \(\varepsilon \rightarrow 0; V^\varepsilon_i(\omega) \rightarrow V_i(\omega)\) in probability.
Theorem (Froyland, González-Tokman, Q)

Let \((\Omega, \mathbb{P})\) be a measure space; \(\sigma\) an ergodic invertible measure-preserving transformation. Let \(A_\omega\) be a measurable family of operators on \(l^2\) with the \((i, j)\) entry \(o(2^{-(i+j)})\).

Let the Lyapunov exponents be \(\lambda_1 > \lambda_2 \ldots\) be the Lyapunov exponents of the cocycle \(A^{(n)}(\omega)\).

Then let \(A^\varepsilon(\omega) = A(\omega) + \varepsilon P\) and let \(\lambda_1^\varepsilon > \ldots > \lambda_d^\varepsilon\) be the exponents of the perturbed cocycle.

Then \(\lambda_i^\varepsilon \to \lambda_i\) as \(\varepsilon \to 0\); \(V_i^\varepsilon(\omega) \to V_i(\omega)\) in probability.
“Very nice. Can you do it with Perron-Frobenius operators?”

“Maybe look at expanding analytic maps on the circle?” (M. Urbanski)
A Blaschke product is a map of the form

\[ B(z) = \alpha \prod_{i=1}^{n} \frac{z + a_i}{1 + \bar{a}_i z} \]

with \(|\alpha| = 1; |a_i| < 1\) for each \(i\). These are analytic maps of the unit disk onto the unit disk. They commute with inversion. There are simple sufficient conditions in terms of the \(a\)'s for \(B\) to be an expanding map of the unit circle.

e.g. \(T_0(z) = z^2\) or \(T_1(z) = \left( \frac{z + \frac{1}{4}}{1 + \frac{z}{4}} \right)^2\).
Random Blaschke product composition

We consider random compositions of Blaschke products and their Perron-Frobenius operators.

\[ \mathcal{L}_B f(z) = \sum_{y \in B^{-1}z} \frac{f(y)}{B'(y)}. \]

Assume that \( B(A^c_R) \subset A^c_r \) for some \( r < R < 1 \). These operators act (very) compactly on \( H^2(A_R) \), the analytic functions on the annulus \( A_R \) (norm = \( L^2 \) norm on the boundary). [Proof: Cauchy integral formula, Arzelà-Ascoli].
Let $r < R < 1$. Let $\sigma : (\Omega, \mathbb{P}) \to (\Omega, \mathbb{P})$ be an invertible ergodic measure-preserving transformation. Suppose $B_\omega$ is a Blaschke product mapping $A_c^R$ into $A_c^R$ for all $\omega$. Call this a Blaschke cocycle; and let $\mathcal{L}_\omega$ be the Perron-Frobenius operator of $B_\omega$ acting on $H^2(A_R)$. 
Random Blaschke Composition: random fixed point

Theorem (González-Tokman, Q)

Consider a Blaschke cocycle

1. There exists a ‘random fixed point’: an $x_\omega$ such that $|x_\omega| < r$; $B_\omega(x_\omega) = x_{\sigma\omega}$;

2. The Lyapunov spectrum of the Perron-Frobenius cocycle is precisely
   - $0$ with multiplicity 1; and
   - $-n\Lambda$ with multiplicity 2 for each $n \in \mathbb{N}$, where $\Lambda = \int \log |B'_\omega(x_\omega)| \, d\mathbb{P}(\omega)$.

Item 2. may be seen as a random version of a theorem of Bandtlow, Just and Slipantschuk, analysing the spectrum of a single Blaschke Perron-Frobenius operator.
Set $B_0(z) = z^2$ and $B_1(z) = \left(\frac{z+1/4}{1+z/4}\right)^2$; $\mathcal{L}_0, \mathcal{L}_1$ the corresponding Perron-Frobenius operators.

Set $\Omega = \{0, 1\}^\mathbb{Z}$ with the shift map; and $\mathbb{P}_p$ the Bernoulli measure with $\mathbb{P}_p([0]) = p$. Define

$$
B_\omega = \begin{cases} 
B_0 & \text{if } \omega_0 = 0; \\
B_1 & \text{if } \omega_0 = 1.
\end{cases}
$$

Then for $p < \frac{1}{2}$, $\Lambda > -\infty$; The Lyapunov spectrum of the P-F cocycle is $\lambda_1, \lambda_2, \ldots = 0, \Lambda, 2\Lambda, \ldots$.

For $p \geq \frac{1}{2}$, $\Lambda = -\infty$; The Lyapunov spectrum of the P-F cocycle is $\lambda_1 = 0$; $\lambda_n = 0$ for all $n > 1$. 
Let $B_i$, $\mathcal{L}_i$ and $\mathbb{P}_p$ be as before. Let $B^\epsilon_i$ be the random map $R_{\epsilon N} \circ B_i$ ($N$ is a standard normal random variable); and $\mathcal{L}_i^\epsilon$ its (annealed) P-F operator (i.e. the average over the P-F operators with the various rotations).

Then for $p \geq \frac{1}{4}$, $\lambda_1^\epsilon = 0$; $\lambda_n^\epsilon = -\infty$ for all $n > 1$.

For $p < \frac{1}{4}$, $\lambda_n^\epsilon > -\infty$ for all $n$?

In particular for $n > 1$ and $p \in [\frac{1}{4}, \frac{1}{2})$, $\lambda_n > -\infty$; $\lambda_n^\epsilon = -\infty$ for all $\epsilon > 0$. 
Theorem (G-T,Q)

If \( T'_\omega(x_\omega) \) is uniformly bounded away from 0, and \( \|L_\epsilon^\omega - L_\omega\| \to 0 \), then for each \( n \), \( \lambda^\epsilon_n \to \lambda_n \) as \( \epsilon \to 0 \).

If \( T'_\omega(x_\omega) \) is not uniformly bounded away from 0, then there exist perturbations \( L_\epsilon^\omega \) such that \( \|L_\epsilon^\omega - L_\omega\| \leq \epsilon \) for all \( \omega \) with \( \lambda_2 = -\infty \).
Thanks!