Quantitative equidistribution of horocycle push-forwards of transverse arcs

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Let

- $\Gamma$ be a co-compact lattice in $\text{PSL}(2, \mathbb{R})$,
- $M = \Gamma \backslash \text{PSL}(2, \mathbb{R}) = T^1(S)$, where $S = \Gamma \backslash \mathbb{H}$,
- $\mu$ be the probability measure on $M$ induced by Haar,
- $\mathfrak{sl}_2(\mathbb{R})$ be the Lie algebra of $\text{PSL}(2, \mathbb{R})$.

For any $W \in \mathfrak{sl}_2(\mathbb{R}) \setminus \{0\}$, the homogeneous flow induced by $W$ is

$$\varphi^W_t(\Gamma g) = \Gamma g \exp(tW),$$
Homogeneous flows

On $M = \Gamma \backslash \text{PSL}(2, \mathbb{R})$, for $p = \Gamma g \in M$,

- geodesic flow:

$$
\varphi_t^X(p) = p \exp(tX) = p \begin{pmatrix}
  e^{t/2} & 0 \\
  0 & e^{-t/2}
\end{pmatrix}, \quad \text{where} \quad X = \begin{pmatrix}
  1/2 & 0 \\
  0 & -1/2
\end{pmatrix},
$$

- (stable) horocycle flow:

$$
h_t(p) := \varphi_t^U(p) = p \exp(tU) = p \begin{pmatrix}
  1 & t \\
  0 & 1
\end{pmatrix}, \quad \text{where} \quad U = \begin{pmatrix}
  0 & 1 \\
  0 & 0
\end{pmatrix},
$$

- unstable horocycle flow:

$$
\varphi_t^V(p) = p \exp(tV) = p \begin{pmatrix}
  1 & 0 \\
  t & 1
\end{pmatrix}, \quad \text{where} \quad V = \begin{pmatrix}
  0 & 0 \\
  1 & 0
\end{pmatrix}.
$$
Horocycle flow on $M$:

$$h_t(p) = p \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Properties:

- zero entropy (Gurevic, 1961),
- minimal (Hedlund, 1936),
- uniquely ergodic (Furstenberg, 1973),
- mixing of all orders (Marcus, 1978),
Quantitative mixing

**Theorem (Ratner, 1987)**

Let \( r > 11/2 \). There exist \( 0 \leq \nu_0 < 1 \) and a constant \( C > 0 \) such that for all \( f, g \in W^r(M) \), for all \( t \geq 1 \), \((up to a logarithmic factor)\) we have

\[
\left| \int_M f \circ h_t \cdot g \, d\mu - \left( \int_M f \, d\mu \right) \left( \int_M g \, d\mu \right) \right| \leq C \|f\|_r \|g\|_r t^{-(1-\nu_0)}.
\]

Let \( \mu_0 > 0 \) be the smallest positive eigenvalue of the Laplace-Beltrami operator on \( S \) (where \( M = T^1(S) \)). Then,

\[
\nu_0 := \begin{cases} \sqrt{1 - 4\mu_0} & \text{if } \mu_0 < 1/4, \\ 0 & \text{if } \mu_0 \geq 1/4. \end{cases}
\]
For any $p \in M$, and $t, s \in \mathbb{R}$,

$$h_t \circ \phi_s^X(p) = \phi_s^X \circ h_{e^s t}(p).$$
Mixing via shearing

\[ q_t := h_t(p) \approx h_{s\tau}(q_t) \]
Mixing via shearing
Let $\alpha: M \to \mathbb{R}_{>0}$ be sufficiently smooth. The time-change $\{\tilde{h}_t\}_{t \in \mathbb{R}}$ defined by $\alpha$ is the smooth flow generated by $\tilde{U} = \alpha^{-1}U$.

- they preserve the smooth measure $\alpha \, d\mu$,
- they are ergodic,
- generic time-changes are not measurably conjugated to the horocycle flow (Ratner, 1986 with Flaminio-Forni, 2003).
Time-changes

Since $\mathcal{L}_U(X) = [U, X] = -U$, we have

$$\mathcal{L}_{\tilde{U}}(X) = [\tilde{U}, X] = -\tilde{U} - (X \alpha^{-1}) U = -\left(1 - \frac{X \alpha}{\alpha}\right) \tilde{U}.$$  

If, for example, $\|X \alpha\|_{\infty} < 1$ (Kushnirenko condition), there is a non-zero (infinitesimal) shear of geodesic segments in direction $\tilde{U}$. 

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Some results:

- mixing (Kushnirenko, 1974, Marcus, 1977),
- Lebesgue spectrum and polynomial mixing (Forni-Ulcigrai, 2012),
- polynomial 3-mixing (Kanigowski-R., 2019).

An analogous approach has been used to study locally Hamiltonian flows on surfaces and time-changes of nilflows.
Back to the horocycle flow

For \( f, g \in C^1(M) \), \( f \in L^2_0(M) \),

\[
\langle f \circ h_t, g \rangle = \int_0^1 \langle f \circ h_t \circ \phi^X_s, g \circ \phi^X_s \rangle \, ds
\]

\[
= \langle \int_0^1 f \circ h_t \circ \phi^X_s \, ds, g \circ \phi^X_1 \rangle - \int_0^1 \langle \int_0^\sigma f \circ h_t \circ \phi^X_s \, ds, Xg \circ \phi^X_s \rangle \, d\sigma,
\]

so that

\[
|\langle f \circ h_t, g \rangle| \leq \|g\|_2 \left\| \int_0^1 f \circ h_t \circ \phi^X_s \, ds \right\|_2 + \|Xg\|_2 \sup_{\sigma \in [0,1]} \left\| \int_0^\sigma f \circ h_t \circ \phi^X_s \, ds \right\|_2.
\]
Goal: let $f$ be sufficiently regular and with zero average, we estimate

$$[(h_t)_* \gamma^X_{p,\sigma}](f) := \int_0^\sigma f \circ h_t \circ \varphi^X_s(p) \, ds$$

for all fixed $p \in M$ and $\sigma \geq 0$.

Since

$$\frac{d}{ds} (h_t \circ \varphi^X_s)(p) = \text{Ad}_{\exp(tU)}(X) = X + tU,$$

we can write

$$[(h_t)_* \gamma^X_{p,\sigma}](f) = \frac{1}{t} \int_{h_t \circ \gamma^X_{p,\sigma}} f \, \hat{U}$$
Deviations of ergodic averages

Theorem (Flaminio-Forni, 2003 - simplified version)

Let $f$ be a smooth function with zero average. There exists a constant $C = C(f)$ such that for all $p \in M$ and for all $t > 0$,

$$\left| \int_0^t f \circ h_s(p) \, ds \right| \leq C t^{\frac{1+v_0}{2}}.$$ 

Hence, we get $|\langle f \circ h_t, g \rangle| \leq C(f, g) t^{-\frac{1-v_0}{2}}$.

**Problem:** the bound in Ratner’s result is of the form $|\langle f \circ h_t, g \rangle| \leq C(f, g) t^{-(1-v_0)}$. 

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Equidistribution of sheared arcs

Idea: replace the geodesic segments with other homogeneous segments $\gamma^W_{p,\sigma}(s) = \varphi^W_s(p)$ for $W \in \mathfrak{sl}_2(\mathbb{R}) \setminus \{0\}$.

**Theorem (consequence of Bufetov-Forni, 2014)**

Let $r > 11/2$, and let

$$W = xX + uU \in \mathfrak{sl}_2(\mathbb{R}),$$

with $x \neq 0$.

For every $\sigma > 0$, there exists a constant $C = C(r, \sigma) > 0$ such that for all $f \in W^r(M) \cap L^2_0(M)$, for all $t \geq 1$, for all $0 \leq S \leq \sigma$, for all $p \in M$, (up to a logarithmic factor) we have

$$\left| \int_0^S f \circ h_t \circ \varphi^W_s(p) \, ds \right| \leq C \| f \|_{r} t^{-\frac{(1-v_0)}{2}}.$$
Main result

**Theorem (R, 2019)**

Let \( r > 11/2 \) and let

\[
W = vV + xX + uU \in \mathfrak{sl}_2(\mathbb{R}),
\]

with \( v \neq 0 \).

For every \( \sigma > 0 \), there exists a constant \( C = C(r, \sigma, W) > 0 \) such that for all \( f \in W^r(M) \cap L^2_0(M) \), for all \( t \geq 1 \), for all \( 0 \leq S \leq \sigma \), for all \( p \in M \), (up to a logarithmic factor) we have

\[
\left| \int_0^S f \circ h_t \circ \varphi^W_s(p) \, ds \right| \leq C \| f \|_r t^{-(1-v_0)}.
\]
Corollary and open questions

**Corollary:** Ratner’s mixing rates follow by choosing, e.g., $W = V$.

**Q.1:** Is the same result true for smooth time-changes of the horocycle flow?

**Q.2:** Can one get sharper estimates for the decay of correlations for smooth time-changes of the horocycle flow?
Sheared curves

Consider $\mathcal{W} = V$, denote $\gamma = \gamma_{p, \sigma}^V$ the curve $\gamma(s) = \varphi_s^V(p)$ for $s \in [0, \sigma]$. Then

$$h_t \circ \gamma(s) = h_t \circ \varphi_s^V(p) = p \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$\frac{d}{ds} h_t \circ \gamma(s) = \frac{d}{ds} h_t \circ \varphi_s^V(p) = \text{Ad}_{\exp(tU)}(V) = V - tX - t^2 U.$$

Hence,

$$\left| \int_0^\sigma f \circ h_t \circ \varphi_s^V(p) \, ds \right| = \frac{1}{t^2} \int_{h_t \circ \gamma} f \hat{U}$$
Sheared curves

\[ \mathcal{O}(t\sigma) = \]

\[ h_t(\varphi) \]

\[ \mathcal{O}(\sigma) = \]

\[ h_t \circ \varphi \]

\[ = \mathcal{O}(t^2 \sigma) \]
This strategy does not provide the optimal rates.
$L^2$ decomposition

Let

$$\Box = -X^2 - \frac{1}{2} UV - \frac{1}{2} VU$$

be the normalized Casimir operator. We can write

$$L^2(M) = \bigoplus_{\mu \in \text{Spec}(\Box)} H_\mu$$

$$W^r(M) = \bigoplus_{\mu \in \text{Spec}(\Box)} W^r(H_\mu).$$

- For $\mu = 0$, the multiplicity is 1 and $H_0 = \{\text{constants}\}$,
- Discrete series: $\mu = -n^2 + n < 0$, for $n \in \mathbb{N}$,
- Principal and complementary series: $\mu > 0$ are the eigenvalues of the Laplace-Beltrami operator on $S$. 

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Estimates in $W^r(H_\mu)$

For $\mu \in \text{Spec}(\Box)$, let $\nu = \Re(\sqrt{1 - 4\mu})$.

We use different strategies for $\mu > 0$ and $\mu < 0$:

- if $f \in W^r(H_\mu)$, with $\mu > 0$ and $\mu \neq \frac{1}{4}$, we prove
  \[
  \left| \int_0^\sigma f \circ h_t \circ \phi_s^V(p) \, ds \right| \leq C \| f \|_r t^{-(1-\nu)}.
  \]

- if $f \in W^r(H_\mu)$, with $\mu < 0$, we prove
  \[
  \left| \int_0^\sigma f \circ h_t \circ \phi_s^V(p) \, ds \right| \leq C \| f \|_r \frac{\log t}{t}.
  \]
Let

\[ I(H_\mu) = \{ D \in (C^\infty(H_\mu))' : LU D = 0 \}. \]

**Theorem (Flaminio-Forni, 2002)**

- If \( \mu > 0 \), then \( I(H_\mu) \) has dimension 2 and is generated by two eigenvectors \( D^+_\mu, D^-_\mu \) for \( X \).
- If \( \mu < 0 \), then \( I(H_\mu) \) has dimension 1 and is generated by an eigenvector \( D^+_\mu \) for \( X \).

For example, if \( f_\mu \in C^\infty(H_\mu) \) is an eigenfunction of the Laplace-Beltrami operator on \( S \), then \( D^+_\mu(f_\mu) = 1 \) and \( D^-_\mu(f_\mu) = 0 \).
Theorem (Flaminio-Forni, 2003)

Let $\mu > 0$. If $f \in C^\infty(H_\mu)$, then for all $p \in M$ and for all $t > 0$ we have

$$\int_0^t f \circ h_r(p) \, dr \approx D^+(f)t^{\frac{1+\nu}{2}} + D^-(f)t^{\frac{1-\nu}{2}} + \text{l.o.t.}$$

Let $\mu < 0$. If $f \in C^\infty(H_\mu)$, then for all $p \in M$ and for all $t \geq 1$ we have

$$\left| \int_0^t f \circ h_r(p) \, dr \right| \leq C \log t.$$
Let $\mu \in \text{Spec}(\square)$, with $\mu > 0$ and $\mu \neq 1/4$.

**Theorem (Bufetov-Forni, 2014)**

There exist two finitely additive functionals $\beta_{\mu}^{\pm}$ on rectifiable arcs and a constant $C > 0$ such that for every rectifiable arc $\gamma$ and for every $t \in \mathbb{R}$

- if $\gamma = \gamma_1 + \gamma_2$, then $\beta_{\mu}^{\pm}(\gamma) = \beta_{\mu}^{\pm}(\gamma_1) + \beta_{\mu}^{\pm}(\gamma_2)$,
- $\beta_{\mu}^{\pm}(\varphi_{-t}^X \gamma) = \exp\left(\frac{1+\nu}{2} t\right) \beta_{\mu}^{\pm}(\gamma)$,
- $\beta_{\mu}^{\pm}(\varphi_{t}^V \gamma) = \beta_{\mu}^{\pm}(\gamma)$,
- $|\beta_{\mu}^{\pm}(\gamma)| \leq C \left(1 + \int_{\gamma} |\hat{X}| + \int_{\gamma} |\hat{U}| \int_{\gamma} |\hat{V}| \right) \left(\int_{\gamma} |\hat{U}| \right)^{\frac{1+\nu}{2}}$. 


Bufetov-Forni functionals

For $f \in W^r(H_\mu)$, with $r > 11/2$, define

$$\beta_f(\gamma) = D_\mu^+(f)\beta_\mu^+(\gamma) + D_\gamma^-(f)\beta_\mu^-(\gamma).$$

Theorem (Bufetov-Forni, 2014)

For every $r > 11/2$ there exist a constant $C_r > 0$ such that for every $f \in W^r(H_\mu)$ and every rectifiable arc $\gamma$,

$$\left| \int_\gamma f \hat{U} - \beta_f(\gamma) \right| \leq C_r \|f\|_r \left( 1 + \int_\gamma |\hat{X}| + \int_\gamma |\hat{V}| \right).$$
Bufetov-Forni functionals

- Hölder additive functionals asymptotic to ergodic integrals (and related results on limit theorems) have first been obtained for translation flows (Bufetov, 2014).

- In the setting of IETs, these functionals are closely related to limit shapes (Marmi-Moussa-Yoccoz, 2010).

- Recently, they have been constructed also for Heisenberg nilflows (Forni-Kanigowski, 2017).
Proof of the main result

Recall that we need to bound

\[ \int_0^\sigma f \circ h_t \circ \varphi_s^V(p) \, ds. \]

Denoting by \( \gamma = \gamma_{\sigma,p}^V \) the curve \( \gamma(s) = \varphi_s^V(p) \) for \( s \in [0, \sigma] \), recall that we computed

\[ \frac{d}{ds} \gamma(s) = V - tX - t^2 U, \]

so that

\[ \left| \int_0^\sigma f \circ h_t \circ \varphi_s^V(p) \, ds \right| = \frac{1}{t^2} \left| \int_{h_t \circ \gamma} f \widehat{U} \right|. \]

By the Bufetov-Forni Theorem, we can bound

\[ \left| \int_{h_t \circ \gamma} f \widehat{U} - \beta_f(h_t \circ \gamma) \right| \leq C_r \|f\|_r (1 + \sigma + \sigma t). \]
Proof of the main result

We need to bound $|\beta_f(h_t \circ \gamma)|$.

Key observation:

\[
\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{t} & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{t} & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix} \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}
\]

that is,

\[
\varphi^V_{-t} \circ \varphi^X_{2 \log t} \circ h_t \circ \gamma(s) = h_{-s}(p_t), \quad \text{where} \quad p_t = \varphi^V_{-t} \circ \varphi^X_{2 \log t} \circ h_t(p).
\]

In particular,

\[
\frac{d}{ds} \varphi^V_{-t} \circ \varphi^X_{2 \log t} \circ h_t \circ \gamma(s) = \frac{d}{ds} h_{-s}(p_t) = -U.
\]
Proof of the main result

The components of the tangent vector of the curve \( \varphi^{-t}_V \circ \varphi^X_{2\log t} \circ h_t \circ \gamma \) are uniformly bounded in \( t \). Hence

\[
\left| \beta_f (\varphi^{-t}_V \circ \varphi^X_{2\log t} \circ h_t \circ \gamma) \right| \leq C' \| f \| \sigma^{1+v} t^{1+v}.
\]

Finally, by the properties of the Bufetov-Forni functionals

\[
\left| \beta_f (\varphi^{-t}_V \circ \varphi^X_{2\log t} \circ h_t \circ \gamma) \right| = \left| \beta_f (\varphi^X_{2\log t} \circ h_t \circ \gamma) \right| \leq C' \| f \| t^{1+v} | \beta_f (h_t \circ \gamma) | .
\]

The proof is complete.