On topological models of zero entropy loosely Bernoulli systems

F. García-Ramos

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(j.w. with Dominik Kwietniak)
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- There are many connections and parallels between the theory of MPSs and TDSs (ergodic theory and topological dynamics).
Basic relationships

- If we start with a TDS \((X, T)\), there exists a Borel \(T\)-invariant probability measure \(\mu\) (Krylov-Bogoliubov), and hence we obtain a MPS \((X, \mu, T)\).
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- If we start with a MPS \((X, \Sigma, \mu, T)\), there exists a uniquely ergodic TDS \((X', \mu', T')\) isomorphic to \((X, \Sigma, \mu, T)\) (Jewett-Krieger).
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- In this case we say \((X', T')\) is a **topological model for** \((X, \Sigma, \mu, T)\) (uniquely ergodic).
Analogous properties
Relationship

- Analogous properties
  - **Measurable**
    - Entropy
    - Mixing
  - Discrete spectrum on $L^2(X, \mu)$
    - Measure distal
    - $K$-system
  - **Topological**
    - Topological entropy
    - Topologically mixing
    - Discrete spectrum on $C(X)$
    - Distal
    - Completely positive top. entropy
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E.g. any topological model of a mixing MPS is topologically mixing, but there exists uniquely ergodic top. mixing TDS \((X, T)\) such that \((X, \mu, T)\) is not mixing (in some cases even discrete spectrum).
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In this talk we will give a topological "version" of zero entropy loosely Bernoulli systems, to give an answer to this question.

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- An ergodic MPS is Kronecker if and only if the (induced) Koopman operator on $L^2(X, \mu)$ has discrete spectrum (Halmos - von Neumann).
- Two Kronecker MPSs are isomorphic if and only if they are spectrally isomorphic (von Neumann).
von Neumann asked if the uniform Bernoulli measure on $\{0, 1\}^\mathbb{Z}$ is isomorphic to the uniform Bernoulli measure on $\{0, 1, 2\}^\mathbb{Z}$. This question was answered by Kolmogorov using entropy (\(\log_2 6 = \log_3\)). Then, Ornstein proved that Bernoulli systems with the same entropy are always isomorphic. In order to understand the isomorphism class of Bernoulli systems, Ornstein introduced the notions of \(\ldots\)nitely determined and \(\ldots\)very weak Bernoulli. At the heart of these definitions lies the Hamming (\(d\)) metric on words $d(x_1 \ldots x_n, y_1 \ldots y_n) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{x_i \neq y_i}$.
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This metric is also used in information theory as a way to measure "mistake" noise.
Informally, a process is *very weak Bernoulli*, if the process obtained by fixing some letters is very similar (in a $d$ sense) to the original one.

Now take the periodic measure $\mu = \delta(01)\mathbb{N}/2 + \delta(10)\mathbb{N}/2$. Here typical words that start with 0 will be very different to words that begin with 1.
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Other isomorphisms theorems?

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Ornstein-Rudolph-Weiss approached the isomorphism problem from a different perspective.
The edit ($\bar{f}$) metric is defined as follows

$$\bar{f}(x_1 \ldots x_n, y_1 \ldots y_n) = 1 - \frac{k}{n},$$

where $k$ is the largest integer such that for some

$$1 \leq i(1) < i(2) < \ldots < i(k) \leq n$$

and

$$1 \leq j(1) < j(2) < \ldots < j(k) \leq n$$

we have that $x_{i(s)} = y_{j(s)}$ for $s = 1, \ldots, k$. 

The edit metric can measure "noise" that can delete or insert symbols. For example

$$\bar{f}(01010, 00101) = \frac{1}{6},$$

and

$$d(010101, 101010) = 0.$$
The edit ($\tilde{f}$) metric is defined as follows

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For example $\bar{f}(010101, 001010) = 1 - 5/6 = 1/6$, and $d(010101, 101010) = 0$. 
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- Actually, they proved that if $0 < h_\mu(T), h_\mu(T'), < \infty$ then they are also Kakutani equivalent (and the $A$ can be taking arbitrarily large).
- It is easy to see that (very weak) Bernoulli systems are loosely Bernoulli (because $\bar{f} \leq \bar{d}$)
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Zero entropy loosely Bernoulli

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We will now describe topological models for loosely Kronecker systems.
Dynamical pseudometrics

Let \((X, T)\) be a TDS (with metric \(d\)). We say \(\rho\) is a dynamical pseudo-metric on \(X\), if \(\rho(x, y) = \rho(Tx, Ty)\).

Even though Kronecker systems are very well understood from a measurable point of view, only recently we are understanding the range of topological behaviours that the models can have (Downarowicz, GR, Glasner, Jägger, Li, Thouvenot, Ye). More on this later.

The Besicovitch pseudometric is particularly useful for understanding some families of the models for Kronecker MPSs.
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- We define the **Besicovitch pseudometric** as

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- So, if one can consider the Besicovitch pseudometric as an infinite topological Hamming distance.
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Maybe to approach topological models of LK, one could try to use some "Besicovitch" version of the $\bar{f}$.
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- For $S \subset \mathbb{N}$, $\overline{D}(S) = \limsup_{n \to \infty} |\{S \cap \{1, ..., n\}\}| / n$ is the upper density.
- We define the **Feldman-Katok** pseudometric as $\rho_{FK}(x, y) = \inf \{\delta > 0 : \exists$ $\delta$-matched $S, S'$ with $\overline{D}(S'), \overline{D}(S) \geq 1 - \delta \}$. 
Theorem

**Theorem 1 (GR-Kwietniak)** Let $(X, T)$ be a TDS and $\mu$ be an ergodic $T$-invariant Borel probability measure. Then $(X, \mu, T)$ is loosely Kronecker if and only if there exists a Borel set $M \subset X$ with $\mu(M) = 1$ such that $\rho_{FK}(x, y) = 0$ for every $x, y \in M$. This gives a hybrid characterization of when an ergodic Borel measure is loosely Kronecker.

We say $(X, T)$ is topologically loosely Kronecker if $\rho_{FK}(x, y) = 0$ for every $x, y \in M$.

**Theorem 2 (GR-K)** Let $(X, T)$ be a TDS. $(X, T)$ is topologically loosely Kronecker if and only if it is uniquely ergodic and $(X, \mu, T)$ is loosely Kronecker. This gives a purely topological characterization of topological models of loosely Kronecker systems.

F. García-Ramos (j.w. with Dominik Kwietniak)
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**MPS**

Zero entropy loosely Bernoulli $\iff$ topologically loosely Kronecker

Uniquely ergodic TDS
Interplay with topological dynamics

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Comparison

Theorem (GR) Let $(X, T)$ be a TDS and $\mu$ be an ergodic $T$-invariant Borel probability measure. $(X, \mu, T)$ is Kronecker if and only if for every $\tau > 0$ there exists a Borel set $M \subset X$ with $\mu(M) \geq 1 - \tau$ such that $T|_M$ is mean equicontinuous (GR).

The ergodicity hypothesis can be dropped (Huang-Li-Thouvenot-Ye). Here we take large but not full measure sets $M$.

Theorem (Li-Tu-Ye, Downarowicz-Glasner, Fuhrmann-Groger-Lenz). $(X, T)$ is mean equicontinuous if and only if $(X, \mu, T)$ has discrete spectrum on $L^2(X, \mu)$ with continuous eigenfunctions (Li-Tu-Ye, Downarowicz-Glasner, Fuhrmann-Groger-Lenz).

For Besicovitch studying $\rho_B(x, y) = 0$ is too strong.

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**Theorem** (Kerr-Li) Every null TDS (zero top. sequence entropy) is tame.

We have the following (strict) hierarchy for minimal TDS.

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**Proof of claim:** Since $(X, \mu, T)$ is LK there exists a compact abelian group $G$ and $g \in G$, so that $(G, \nu, R)$ is Kakutani equivalent to $(X, \mu, T)$, where $\nu$ is the Haar measure on $G$ and $Rx = g \cdot x$ (isometry).
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Let $\varepsilon > 0$. By ORW, there exists a Borel set $B \subset G$ with $\mu_h(B) \geq 1 - \varepsilon/2$, such that $(X, \mu, T)$ is isomorphic to $(A, \nu_A, R_A)$, using $\phi : X \to A$. 
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- By Lusin’s theorem there exists a compact sets $M \subset X$ with $\mu(M) \nu(A) \geq 1 - \varepsilon / 3$ such that $\phi|_M : M \to \phi(M)$ is (uniformly) continuous.
Proof

Let \( \varepsilon > 0 \). By ORW, there exists a Borel set \( B \subset G \) with \( \mu_h(B) \geq 1 - \varepsilon / 2 \), such that \( (X, \mu, T) \) is isomorphic to \( (A, \nu_A, R_A) \), using \( \phi : X \to A \).

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There exists \( \delta > 0 \) is such that if \( x, y \in M \) and \( d(x, y) \leq \delta \) then \( d(\phi^{-1} R^n \phi(x), \phi^{-1} R^n \phi(y)) \leq \varepsilon \) for every \( n \in \mathbb{N} \) with \( R^n \phi(x), R^n \phi(x) \in \phi(M) \) (1).
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- Now, let $Y_M \subset \phi(M)$ be the set of points in $\phi(M)$ which are $\nu$-generic for $\phi(M)$ with respect to the map $S$. 
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- Thus, we have $\nu(Y_M) = \nu(\phi(M))$, so there is $z \in M$ such that 
  \[ \mu(B_{\delta/2}(z) \cap M \cap \phi^{-1}(Y_M)) > 0. \]
Let $x, y \in B_{\delta/2}(z) \cap M \cap \phi^{-1}Y_M$ (this is our $B$). We will show that $\rho_{FK}(x, y) \leq \varepsilon$. 
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**Strategy:** find a $\delta$-match for the $R_A$ orbits of $\phi(y)$ and $\phi(z)$ that only matches points on $\phi(Y_M)$. 

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F. García-Ramos ()
Sketch

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- Use (1) to get $\varepsilon$-match for the $T$ orbits of $x$ and $y$. 
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- Strategy: find a \( \delta \)-match for the \( R_A \) orbits of \( \phi(y) \) and \( \phi(z) \) that only matches points on \( \phi(Y_M) \).
- Use (1) to get \( \varepsilon \)-match for the \( T \) orbits of \( x \) and \( y \).
- (see picture)
We define

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and

\[ S := c_x \left\{ E_M(x, y) \right\} , \text{ and} \]
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One can check that for every \( i \in S \) there exists \( j \) such that
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d_G(R_B^i \phi(x), R_B^{\pi(i)} \phi(y)) = d_G(R_B^j \phi(x), R_B^j \phi(y)) = d(\phi(x), \phi(y)) \leq \delta'.
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With this we obtain a $\delta'$-match for the $R_A$ orbits of $\phi(y)$ and $\phi(z)$, matches points on $\phi(M)$. 

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\overline{D}(S), \overline{D}(S') \geq \overline{D}(E_M(y, z)) \geq 1 - \varepsilon.
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Thus $\rho_{FK}(y, z) \leq \varepsilon$ (we finish the claim).
Now we can use the fact that \( \rho_{FK}(x, Tx) = 0 \), to prove that there exists a Borel set \( M_\varepsilon \subset X \) with \( \mu(M_\varepsilon) = 1 \) such that \( \rho_{FK}(x, y) \leq \varepsilon \) for every \( x, y \in M_\varepsilon \).
Proof

Now we can use the fact that $\rho_{FK}(x, Tx) = 0$, to prove that there exists a Borel set $M_\varepsilon \subset X$ with $\mu(M_\varepsilon) = 1$ such that $\rho_{FK}(x, y) \leq \varepsilon$ for every $x, y \in M_\varepsilon$.

We conclude the result.
Proof

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- We conclude the result.

- Note that if we assume that the map is uniquely ergodic we do not get directly that $\rho_{FK}(x, y) = 0$ for every $x, y \in M$. 

Proof

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- We conclude the result.
- Note that if we assume that the map is uniquely ergodic we do not get directly that $\rho_{FK}(x, y) = 0$ for every $x, y \in M$.
- This proof has to be done with a different approach.