Stabilized automorphism groups and local $P$ entropy

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1. Stabilized automorphism groups
2. Local $\mathcal{P}$ entropy
3. Local $PS$ entropy for stabilized groups
4. Distinguishing stabilized automorphism groups of SFT’s
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4. Distinguishing stabilized automorphism groups of SFT’s
Let $T: X \rightarrow X$ be a homeomorphism of a compact metric space.
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An *automorphism* of $T : X \to X$ is a homeomorphism $\phi : X \to X$ such that $\phi T = T \phi$. 
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The collection of all automorphisms of $T: X \to X$ forms a group denoted Aut($T$).
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Let \( (X_n, \sigma_n) = \text{full shift on } n \text{ symbols} \). \( \text{Aut}(\sigma_n) \) first studied by Hedlund and others in late 60's.
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- \( (Y, \sigma_Y) = \text{subshift from Thue-Morse sequence} \implies \text{Aut}(\sigma_Y) \cong \mathbb{Z} \oplus \mathbb{Z}/2. \)

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- $(Z, \sigma_Z) = \text{mixing shift of finite type} \implies \text{Aut}(\sigma_Z) \text{ large}$. 

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$\text{Aut}(\sigma_n)$ first studied by Hedlund and others in late 60’s.
Despite many years, $\text{Aut}(\sigma_n)$ is still quite mysterious. For example:

- What is the abelianization $\text{Aut}(\sigma_n)_{ab}$?
- For which $m, n$ is $\text{Aut}(\sigma_m)$ isomorphic to $\text{Aut}(\sigma_n)$?

We know $\text{Aut}(\sigma_2)$ is not isomorphic to $\text{Aut}(\sigma_4)$.

This is an exercise using: Theorem (Ryan's Theorem)

The center of $\text{Aut}(\sigma_n)$ is generated by $\sigma_n$.

But, for example, we can't distinguish any two of: $\text{Aut}(\sigma_2), \text{Aut}(\sigma_3), \text{Aut}(\sigma_6)$.
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For a system \((X, T)\) where \(T: X \to X\) is a homeomorphism, \(X\) compact metric, the *stabilized automorphism group* is

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\text{Aut}^{(\infty)}(T) = \bigcup_{k=1}^{\infty} \text{Aut}(T^k).
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If \((X, \sigma)\) is a non-trivial SFT, then \(\text{Aut}(\sigma^k) \subset \text{Aut}(\sigma^{mk})\) is proper for any \(k \geq 1, m \geq 2\).
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is in \(\text{Aut}(\sigma^2_X)\) but not \(\text{Aut}(\sigma_2)\).
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Some properties

- For any system \((X, T)\) and \(k \geq 1\), \(\text{Aut}^{(\infty)}(T) = \text{Aut}^{(\infty)}(T^k)\).

- If \((X, T)\) and \((Y, S)\) are topologically conjugate, then \(\text{Aut}^{(\infty)}(T) \cong \text{Aut}^{(\infty)}(S)\).
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It follows that:

**Proposition**

*Let \(m, n \geq 2\). If \(n^k = m^j\), then*

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$$GL_k(\mathcal{R})_{ab} =??$$
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Whitehead, 1950: consider \( GL(\mathcal{R}) = \bigcup_{k=1}^{\infty} GL_k(\mathcal{R}) \). Then

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but we understand \([Aut^{(\infty)}(\sigma_n), Aut^{(\infty)}(\sigma_n)]\).
For a mixing SFT defined by a primitive $\mathbb{Z}_+\text{-matrix } A$ there is a homomorphism called the dimension representation

$$\pi_A : \text{Aut}(\sigma_A) \rightarrow \text{Aut}(G_A, G_A^+, \delta_A)$$

where:

- $G_A$ is the dimension group associated to $A$
- $G_A^+$ is a positive cone in $G_A$
- $\delta_A$ is a distinguished automorphism of $G_A$.

For full shifts, the dimension representation is a surjective homomorphism

$$\pi_n : \text{Aut}(\sigma_n) \rightarrow \mathbb{Z}^\omega(n)$$

where $\omega(n) = \# \text{distinct primes dividing } n$.

Note: $\pi_A$ is defined using dynamical data, not group theoretically!

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**Theorem (Hartman-Kra-S.)**

The dimension representation $\pi_n : \text{Aut}(\sigma_n) \rightarrow \mathbb{Z}^{\omega(n)}$ extends to a ‘stabilized’ dimension representation

$$\pi_n^{(\infty)} : \text{Aut}^{(\infty)}(\sigma_n) \rightarrow \mathbb{Z}^{\omega(n)}$$

and this homomorphism $\pi^{(\infty)}$ is the abelianization of $\text{Aut}^{(\infty)}(\sigma_n)$. 
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And as a corollary, we get:

**Corollary (Hartman-Kra-S.)**

*If* $\text{Aut}^{(\infty)}(\sigma_m)$ *is isomorphic to* $\text{Aut}^{(\infty)}(\sigma_n)$, *then* $\omega(m) = \omega(n)$. *So, for example,*

$$\text{Aut}^{(\infty)}(\sigma_6) \not\cong \text{Aut}^{(\infty)}(\sigma_2).$$
Let $\text{Inert}^{(\infty)}(\sigma_n) = \ker \pi_n^{(\infty)}$. So $\text{Inert}^{(\infty)}(\sigma_n) = \bigcup_{k=1}^{\infty} \text{Inert}(\sigma_n^k)$. The previous theorem says that $\text{Inert}^{(\infty)}(\sigma_n)$ is the commutator subgroup of $\text{Aut}^{(\infty)}(\sigma_n)$. We also proved:

**Theorem (Hartman-Kra-S.)**

For any $n \geq 2$, the group $\text{Inert}^{(\infty)}(\sigma_n) = \bigcup_{k=1}^{\infty} \text{Inert}(\sigma_n^k) = \ker \pi_n^{(\infty)}$ is simple.

Note the group $\text{Aut}^{(\infty)}(\sigma_n)$, and hence also $\text{Inert}^{(\infty)}(\sigma_n)$, is residually finite, and thus very far from simple!
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In summary, $\text{Aut}^{(\infty)}(\sigma_2)$ and $\text{Aut}^{(\infty)}(\sigma_6)$ are not isomorphic because they have different abelianizations:

$$\text{Aut}^{(\infty)}(\sigma_2)_{ab} \cong \mathbb{Z}, \quad \text{Aut}^{(\infty)}(\sigma_6)_{ab} \cong \mathbb{Z}^2.$$
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But what about $\text{Aut}^{(\infty)}(\sigma_2)$ and $\text{Aut}^{(\infty)}(\sigma_3)$?
In summary, $\text{Aut}^{(\infty)}(\sigma_2)$ and $\text{Aut}^{(\infty)}(\sigma_6)$ are not isomorphic because they have different abelianizations:

$$\text{Aut}^{(\infty)}(\sigma_2)_{ab} \cong \mathbb{Z}, \quad \text{Aut}^{(\infty)}(\sigma_6)_{ab} \cong \mathbb{Z}^2.$$ 

But what about $\text{Aut}^{(\infty)}(\sigma_2)$ and $\text{Aut}^{(\infty)}(\sigma_3)$? Each of them are extensions of $\mathbb{Z}$ by a simple group:

$$\text{Aut}^{(\infty)}(\sigma_2) \cong \text{Inert}^{(\infty)}(\sigma_2) \rtimes \mathbb{Z}$$

$$\text{Aut}^{(\infty)}(\sigma_3) \cong \text{Inert}^{(\infty)}(\sigma_3) \rtimes \mathbb{Z}.$$
1. Stabilized automorphism groups

2. Local $\mathcal{P}$ entropy

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4. Distinguishing stabilized automorphism groups of SFT’s
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- choose a class \(\mathcal{P}\) of finite groups which is closed under isomorphism, e.g. finite abelian groups.
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- choose a class \(\mathcal{P}\) of finite groups which is closed under isomorphism, e.g. finite abelian groups.
- try to approximate \(C(G^n)\)'s with finite \(\mathcal{P}\) groups, compute some growth.
Fix a leveled group \((G, g)\), and some class \(\mathcal{P}\) of finite groups, closed under isomorphism.
Local $\mathcal{P}$ entropy

Fix a leveled group $(G, g)$, and some class $\mathcal{P}$ of finite groups, closed under isomorphism.

A subgroup $H \subset G$ is $g$-locally $\mathcal{P}$ if it satisfies:

1. For all $n$ sufficiently large, $H \cap C(g^n) \in \mathcal{P}$.
2. $H \cap C(g) \neq \{e\}$. 

Define the local $\mathcal{P}$ entropy of $(G, g)$ by

\[ h_{\mathcal{P}}(G, g) = \sup_{H \in \mathcal{F}_\mathcal{P}} \limsup_{n \to \infty} \frac{1}{n} \log \log |H \cap C(g^n)| \]

where $\mathcal{F}_\mathcal{P}$ is the set of $g$-locally $\mathcal{P}$ subgroups of $G$. 

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Local \(\mathcal{P}\) entropy
Some properties

The choice of $\mathcal{P}$ provides some flexibility; a downside is that, depending on $G, g$ and $\mathcal{P}$, the quantity $h_{\mathcal{P}(G,g)}$ may not exist.

1. If $\iota: \langle H, h \rangle \to \langle G, g \rangle$ is a leveled monomorphism, then $h_{\mathcal{P}(H,h)} \leq h_{\mathcal{P}(G,g)}$.

2. If $\Psi: \langle H, h \rangle \to \langle G, g \rangle$ is a leveled isomorphism, then $h_{\mathcal{P}(H,h)} = h_{\mathcal{P}(G,g)}$.

3. For any $k \geq 1$, $h_{\mathcal{P}(G,g)} \leq k \cdot h_{\mathcal{P}(G,g)}$.

4. $h_{\mathcal{P}(G,g)} = h_{\mathcal{P}(G,g^{-1})}$ (a leveled map $\phi: \langle G, g \rangle \to \langle H, h \rangle$ is a group homomorphism $\phi: G \to H$ with $\phi(g) = h$).

Item 2 is particularly important for our applications - it means we can use $h_{\mathcal{P}}$ as an invariant of leveled isomorphism.
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Item 2 is particularly important for our applications - it means we can use $h_{l\mathcal{P}}$ as an invariant of leveled isomorphism.
As an example, let \( p_k \) be the set of primes, \( Q \) be a partition of \( \mathbb{N} \) into sets of size \( p_k \), one set per prime.
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Let $H$ be the group of finitely-supported permutations of $\mathbb{N}$ respecting $Q$, and $G$ be the group generated by $\tau$ and $H$ in $\text{Sym}(\mathbb{N})$. 
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Let $H$ be the group of finitely-supported permutations of $\mathbb{N}$ respecting $Q$, and $G$ be the group generated by $\tau$ and $H$ in $\text{Sym}(\mathbb{N})$.

For any prime $q$, if $\mathcal{P}_q$ is the class of finite abelian $q$-groups, then

$$H_{l\mathcal{P}_q}(G, \tau) = \frac{1}{q} \log q.$$
We can use local $P$ entropy to study stabilized automorphism groups.
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For this, we use the class of finite groups which are products of simple non-abelian groups (also satisfying some technical condition, we omit):

$$PS_r = \{ G \mid G \cong \prod_{i=1}^{r} G_i \text{ for some finite simple non-abelian groups } G_i \}.$$
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So $PS_1$ is just the class of finite simple non-abelian groups. That $PS_r$ is closed under isomorphism follows from the Krull-Remak-Schmidt Theorem.
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One of our main results is the following:

**Theorem (S.)**

Let $(X, T)$ be an expansive system such that for any $a_j \to \infty$, the set of periodic points $\bigcup_{j=1}^{\infty} P_{a_j}(T)$ is dense in $X$. Then for any $k \geq 1$, $r \geq 1$, if the local $PS_r$ entropy of $(\text{Aut}^{(\infty)}(T), T^k)$ exists, then

$$h_{lPS_r} \left( \text{Aut}^{(\infty)}(T), T^k \right) \leq h_{top}(T^k).$$
Our primary application is in the setting of shifts of finite type, to which the previous theorem applies. In this case, we can say much more:

\[ h_{lPS}^r(\text{Aut}(\sigma), \sigma^k) = h_{\text{top}}(\sigma^k). \]
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**Theorem (S.)**

Let $(X, \sigma)$ be a non-trivial mixing shift of finite type, and $k \geq 1$. There exists $r \geq 1$ such that

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- For full shifts, can actually just use the class \(\mathcal{P}\) of all finite simple groups.
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Let $(X, \sigma_X)$ and $(Y, \sigma_Y)$ be non-trivial mixing shifts of finite type, and suppose there is an isomorphism of groups

$$\Psi : \text{Aut}^{(\infty)}(\sigma_X) \rightarrow \text{Aut}^{(\infty)}(\sigma_Y).$$

Then for some $k, j \neq 0$, $\Psi$ is also an isomorphism of leveled groups

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To prove the above, we introduce something called ‘ghost centers’ of a group. While the center of $\text{Aut}^{(\infty)}(\sigma_X)$ is trivial, ghost centers of $\text{Aut}^{(\infty)}(\sigma_X)$ always exist.
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- Note: if \((X, \sigma_X), (Y, \sigma_Y)\) are presented as edge SFT’s with primitive adjacency matrices \(A, B\), then this says
\[
\frac{\log \lambda_A}{\log \lambda_B} \in \mathbb{Q}
\]

where \(\lambda_A, \lambda_B\) are the PF-eigenvalues of \(A, B\).
This allows us to give a complete classification of the stabilized automorphism groups of full shifts:

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Given natural numbers $m, n \geq 2$, the stabilized groups $\text{Aut}(\sigma^m)$ and $\text{Aut}(\sigma^n)$ are isomorphic if and only if $m^k = n^j$ for some $k, j \in \mathbb{N}$.

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The stabilized automorphism groups of the full 2-shift and full 3-shift are not isomorphic.
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1. $G = \prod_{i=1}^{r} G_i$ where each $G_i$ is finite, simple, and non-abelian.
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Now to show $h_{lPS_{C,D,r}} \left( \text{Aut}^{(\infty)}(\sigma_X), \sigma_X^k \right) \geq h_{top}(\sigma_X^k)$, it suffices to produce a $\sigma_X^k$-locally $PS_{C,D,r}$ subgroup with the right growth rate.
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1. $G = \prod_{i=1}^{r} G_i$ where each $G_i$ is finite, simple, and non-abelian.
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Now to show $h_{lPS_{C,D,r}} \left( \text{Aut}^{(\infty)}(\sigma_X), \sigma_X^k \right) \geq h_{\text{top}}(\sigma_X^k)$, it suffices to produce a $\sigma_X^k$-locally $PS_{C,D,r}$ subgroup with the right growth rate.

For simplicity, we’ll outline this for full shifts and $k = 1$, for which we can use $C = D = r = 1$, and $PS_{1,1,1}$ is just the class of finite simple non-abelian groups.
Let $\Gamma_n$ be a graph with one vertex, $n$ labeled edges.

So $(X_n, \sigma_n)$ is the edge SFT coming from $\Gamma_n$. 

\[
\begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{graph.png}}
\end{array}
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We may consider $(X_n, \sigma_n^k)$ as the full shift on vectors of length $k$

$$
\begin{pmatrix}
  a_0 \\
  \vdots \\
  a_{k-1}
\end{pmatrix}, \quad a_i \in \{0, \ldots, n-1\}.
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So a point in $(X_n, \sigma_n^k)$ looks like

$$
\cdots \begin{pmatrix} a_{-k} \end{pmatrix} \begin{pmatrix} a_0 \end{pmatrix} \begin{pmatrix} a_k \end{pmatrix} \cdots
\begin{pmatrix} a_{-1} \end{pmatrix} \begin{pmatrix} a_{k-1} \end{pmatrix} \begin{pmatrix} a_{2k-1} \end{pmatrix} \cdots
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\[
\gamma \in \text{Sym}(E(\Gamma_n)) \mapsto \tilde{\gamma} \in \text{Aut}(\sigma_n)
\]

$\tilde{\gamma}$ given by 0-block code.

Likewise, $\gamma \in \text{Sym}(E(\Gamma_n^k)) \mapsto \tilde{\gamma} \in \text{Aut}(\sigma_n^k)$. 

Define $\text{Simp}(k)(\sigma_n) = \{ \tilde{\gamma} | \gamma \in \text{Sym}(E(\Gamma_n^k)) \}$.

So $\text{Simp}(k)(\sigma_n) \subset \text{Aut}(\sigma_n^k)$. 

Note for every $k$, $\text{Simp}(k)(\sigma_n)$ is a finite group.
Let $E(\Gamma_n) =$ labeled edge set of $\Gamma_n$.

$$\gamma \in \text{Sym}(E(\Gamma_n)) \iff \tilde{\gamma} \in \text{Aut}(\sigma_n)$$

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Such a $\tilde{\gamma}$ is called an elementary simple automorphism (of $(X_n, \sigma_n)$).
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Stabilized simple automorphisms

Let $E(\Gamma_n) = \text{labeled edge set of } \Gamma_n$.

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Likewise,

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so

$$\text{Simp}^{(k)}(\sigma_n) \subset \text{Aut}(\sigma^k_n).$$

Note for every $k$, $\text{Simp}^{(k)}(\sigma_n)$ is a finite group.
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There are inclusions

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\text{Alt}^{(k)}(\sigma_n) \hookrightarrow \text{Alt}^{(kj)}(\sigma_n)
\]

For example: \( \tau : 0 \leftrightarrow 1 \)

\[
\tilde{\tau} \in \text{Simp}^{(1)}(\sigma_2) \quad \sim \quad \tilde{\tau} = \begin{pmatrix} \tau \\ \tau \end{pmatrix} \in \text{Simp}^{(3)}(\sigma_2)
\]
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It suffices then to compute $\limsup_{k \to \infty} \frac{1}{k} \log \log 2^n = \log n$, using e.g. Stirling's.
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It suffices then to compute $\limsup_{k \to \infty} \frac{1}{k} \log \log \frac{1}{2} n^k! = \log(n)$, using e.g. Stirling’s.
To prove that isomorphism of stabilized automorphism groups of mixing SFT’s upgrades to leveled isomorphism, we use the concept of *ghost centers*: 

- A subgroup $H \subset \text{Aut}(\sigma^n)$ is a ghost center if for every finitely generated subgroup $K$, $K \cap H \neq \{e\}$.
- Say a ghost center $H$ is:
  - Maximal if it is not properly contained in any other ghost center.
  - Cyclic if $H$ is a cyclic group.

Maximal cyclic ghost centers are mapped to maximal cyclic ghost centers under an isomorphism. Now one shows that:

$H \subset \text{Aut}(\sigma^n)$ is a maximal cyclic ghost center if and only if $H = \langle \gamma \rangle$ where $\gamma$ is a root of a power of $\sigma^n$. 


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(Answer is no in general; so in particular, it is not true in general that all ghost centers are always conjugate.)
A very rough idea of how to get $h_{lPS} \left( \text{Aut}^{(\infty)}(T), T \right)$ bounded above by $h_{top}(T^k)$:
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- For a $T$-locally $PS_r$ subgroup of $\text{Aut}^{(\infty)}(T)$ $H$, get normal subgroups $H \cap C(T^m) \cap \ker \rho_m = H \cap \text{Aut}(T^m) \cap \ker \rho_m$. 
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- Then need to understand the structure of normal subgroups of a $PS_r$ group.
Thank you again!