DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF MARYLAND  
GRADUATE WRITTEN EXAM  
August 2015  

ALGEBRA (Ph.D. Version)  

Instructions to the student  

a. Answer all six questions; each will be assigned a grade from 0 to 10.  
b. Use a different booklet for each question. Write the problem number and your code number (not your name) on the outside of the booklet.  
c. Keep scratch work on separate pages in the same booklet.  

1. Suppose that  

\[ 1 \to A \to B \xrightarrow{\varphi} C \to 1 \]

is an exact sequence of groups, where \( A \) has order 85 and \( C \) has order 9.  
(a) Let \( S \) be a 3-Sylow subgroup of \( B \). Show that \( g \) maps \( S \) isomorphically to \( C \).  
(b) Show that \( |\text{Aut}(A)| = 64 \) (where \( \text{Aut} \) denotes the automorphism group of a group).  
(c) Show that \( B \) is abelian.  

2. Let \( n \geq 2 \) and consider the \( n \times n \) complex matrices defined by  

\[
M = \begin{pmatrix} 
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 1 & \cdots & 0 \\
1 & 0 & 0 & 0 & \cdots & 0 
\end{pmatrix} \quad N = \begin{pmatrix} 
0 & 1 & 0 & 0 & \cdots & 1 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & 0 & \cdots & 0 
\end{pmatrix}
\]

(These are Jordan blocks with an extra 1 added in the upper right corner or the lower left corner).  
(a) Find the characteristic polynomial and the minimal polynomial of \( M \). (+Hint: The vectors \((1, \zeta, \zeta^2, \ldots)^T\), where \( \zeta^n = 1 \), could be useful.)  
(b) Find the characteristic polynomial and the minimal polynomial of \( N \). (+Hint: \( N^2 \) is the same as the square of the Jordan block obtained by removing the 1 in the upper right corner.)  

3. Let \( R = \mathbb{C}[X,Y] \) and let \( M = (X,Y) \) be the ideal of \( R \) generated by \( X \) and \( Y \).  
(a) Show that the map  

\[ M \otimes_R M \to \mathbb{C} \]

defined by  

\[ \sum_i f_i \otimes g_i \to \sum_i \frac{\partial f_i}{\partial X}(0,0) \frac{\partial g_i}{\partial Y}(0,0) \]

is a well-defined map of vector spaces over \( \mathbb{C} \).  
(b) Show that \( Z = X \otimes Y - Y \otimes X \neq 0 \) in \( M \otimes_R M \).  
(c) Show that there is an element \( r \neq 0 \) in \( R \) such that \( rZ = 0 \) in \( M \otimes_R M \).  

4. Let \( R \) be a commutative ring with \( 1 \neq 0 \).  
(a) Let \( I_1, I_2, \ldots, I_n \) be ideals of \( R \) and let \( P \) be a prime ideal of \( R \) such that  

\[ I_1 \cap I_2 \cap \cdots \cap I_n \subseteq P. \]
Show that $I_i \subseteq P$ for some $i$.
(b) A minimal prime ideal of $R$ is a prime ideal $m$ of $R$ such that whenever $Q$ is a prime ideal with $Q \subseteq m$ then $Q = m$. Show that $R$ contains at least one minimal prime ideal (you may use Zorn's Lemma, or something equivalent).
(c) Suppose there are minimal prime ideals $m_1, m_2, \ldots, m_n$ of $R$ such that
$$m_1 \cap m_2 \cap \cdots \cap m_n = (0).$$
Show that $\{m_1, m_2, \ldots, m_n\}$ is the set of all minimal prime ideals of $R$.

5. Let $Q = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_n$ be a sequence of fields such that $K_{i+1}/K_i$ is Galois of degree 3 for each $i \geq 0$. Show that $Q(\sqrt[3]{2})$ is not contained in $K_n$.

6. Let $\rho : G \to \text{GL}_n(C)$ be a representation of the finite group $G$. Define a representation $\bar{\rho} : G \times (\mathbb{Z}/2\mathbb{Z}) \to \text{GL}_2(C)$ by
$$\bar{\rho}(g, 0) = \begin{pmatrix} \rho(g) & 0 \\ 0 & \rho(g) \end{pmatrix}, \quad \bar{\rho}(g, 1) = \begin{pmatrix} 0 & \rho(g) \\ \rho(g) & 0 \end{pmatrix}.$$
(a) Show that $\bar{\rho}$ is a representation of $G \times (\mathbb{Z}/2\mathbb{Z})$.
(b) If $\pi$ is an irreducible representation of $G$, then $\pi'(g, a) = \pi(g)$ defines an irreducible representation $\pi'$ of $G \times (\mathbb{Z}/2\mathbb{Z})$. Show that the number of times that $\pi$ occurs in $\rho$ equals the number of times that $\pi'$ occurs in $\bar{\rho}$.