1. Let $G$ be a group of order $5 \cdot 11 \cdot 17$.
   (a) Show that $G$ contains an element of order 187.
   (b) Show that if $G$ contains an element of order 55 then $G$ is cyclic.

2. Let $M$ be an $n \times n$ complex matrix and let $R = \mathbb{C}[M]$ be the span over the complex numbers of $I, M, M^2, M^3, \ldots$. Show that $M$ is diagonalizable if and only if the ring $R$ contains no nonzero nilpotent matrices (a nilpotent matrix $N$ is a matrix such that $N^k = 0$ for some positive integer $k$).

3. Let $R$ be a commutative ring with 1 and let $P$ be a prime ideal of $R$. Let $M$ be an $R$-module. Define
   
   \[ N = \{ m \in M \mid \text{there exists } r \in R, r \notin P, \text{ with } rm = 0 \}. \]

   (a) Prove that $N$ is a submodule of $M$.
   (b) Suppose that $N = 0$. Let $m$ be a nonzero element of $M$. Show that $Pm = \{ \pi m \mid \pi \in P \}$ is not equal to $Rm$.
   (c) Suppose that $N = 0$. In addition, assume that $M$ is nonzero and has no proper nonzero submodules. Show that $P$ is a maximal ideal of $R$.

4. (a) Let $R$ be a principal ideal domain and let $M$ be a finitely generated $R$-module. Show that there exists an exact sequence

   \[ 0 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0, \]

   where $P_0$ and $P_1$ are projective $R$-modules.

   (b) Let $R$ be a commutative ring with 1 and let $M$ be an $R$-module. Suppose we have a diagram of $R$-modules with exact rows

   \[
   \begin{array}{ccc}
   P_1 & \xrightarrow{f_1} & P_0 & \xrightarrow{h_0} & M & \xrightarrow{} & 0 \\
   & & \| & & \| \\
   Q_1 & \xrightarrow{g_1} & Q_0 & \xrightarrow{h_0} & M & \xrightarrow{} & 0 \\
   \end{array}
   \]

   where $P_0$ and $P_1$ are projective $R$-modules and $Q_0$ and $Q_1$ are arbitrary $R$-modules. Show that there are homomorphisms $h_0 : P_0 \rightarrow Q_0$ and $h_1 : P_1 \rightarrow Q_1$ such that the resulting diagram commutes.
5. (a) Let \( f(X) \) be a monic irreducible polynomial in \( \mathbb{Q}[X] \), and let \( K \) be a finite Galois extension of \( \mathbb{Q} \). If \( g, h \) are monic irreducible factors of \( f \) in \( K[X] \), show that there exists an automorphism \( \sigma \) of \( K \) over \( \mathbb{Q} \) such that \( g = \sigma(h) \) (applied coefficient-wise).
(b) Give an example where this conclusion is not valid if \( K \) is not Galois over \( \mathbb{Q} \).

6. Let \( G \) be a finite group and let \( \rho : G \to \text{GL}_n(\mathbb{C}) \) be a representation of \( G \). Let \( \chi \) be the character of \( \rho \).
(a) Let \( g \in G \). Show that \( \chi(g^{-1}) \) is the complex conjugate of \( \chi(g) \). *(Hint: What does the diagonalization of \( \rho(g) \) look like?)*
(b) Let \( g \in G \). Suppose that \( g^{-1} \) is conjugate to \( g \) (that is, \( g^{-1} = hgh^{-1} \) for some \( h \in G \)). Show that \( \chi(g) \) is a real number.
(c) Let \( S_m \) be the group of permutations of \( m \) objects. Show that every character of \( S_m \) is real valued.