1. Let $G$ be a finite group and let $p$ be the smallest prime dividing $|G|$. Let $S$ be a Sylow $p$-subgroup of $G$. Suppose that $S$ is cyclic of order $p^n$ for some $n \geq 1$ and is a normal subgroup of $G$. Show that $S$ is in the center of $G$.

2. Let $A$ be an $n \times n$ matrix with entries in a field $K$. Define a bilinear pairing on $K^n$ by $$\langle v, w \rangle = (v^T)Aw,$$
where $v, w \in K^n$ are column vectors, and $v^T$ is the row vector that is the transpose of $v$.
We say that the pairing is non-degenerate if $\langle v, w \rangle = 0$ for all $w$ implies that $v = 0$, and $\langle v, w \rangle = 0$ for all $v$ implies that $w = 0$.
Show that the pairing is non-degenerate if and only if $\det(A) \neq 0$.

3. Let $R$ be an integral domain and let $F$ be its field of fractions. Assume that $R$ has a unique maximal ideal $M$.
(a) Show that $M = \{r \in R \mid r = 0 \text{ or } 1/r \notin R\}$.
(b) Suppose that $R$ has the property that, for each $0 \neq r \in F$, at least one of $r$ and $1/r$ is in $R$.
Show that every finitely generated ideal of $R$ is principal.

4. Let $R$ be a PID and let $M$ be a finitely generated $R$-module. Let $F$ be a field containing $R$. Show that $\text{Hom}_{R}(M, F)$ (that is, $R$-module homomorphisms from $M$ to $F$) and $M \otimes_{R} F$ have the same dimension as vector spaces over $F$.

5. Let $n \geq 5$ and let $A_n$ be the group of even permutations of $n$ objects. You may assume the fact that $A_n$ has no normal subgroups except $1$ and $A_n$.
(a) Show that there is no subgroup $H$ of $A_n$ with $1 < [A_n : H] < n$.
(b) Let $f(x) \in Q[x]$ be an irreducible polynomial of degree $n \geq 5$ and suppose that the splitting field of $f(x)$ has Galois group $A_n$. Let $\alpha$ be a root of $f(x)$ and let $K = Q(\alpha)$. Show that if $F$ is a field with $Q \subseteq F \subseteq K$, then $F = Q$ or $F = K$.

6. Let $G$ be a finite group and let $\rho : G \rightarrow \text{GL}_n(C)$ be a representation of $G$. Define $\bar{\rho} : G \times (Z/2Z) \rightarrow \text{GL}_{2n}(C)$ by $$\bar{\rho}(g, 0) = \begin{pmatrix} \rho(g) & 0 \\ 0 & \rho(g) \end{pmatrix}, \quad \bar{\rho}(g, 1) = \begin{pmatrix} 0 & \rho(g) \\ \rho(g) & 0 \end{pmatrix}.$$Show that $\bar{\rho}$ is a representation of $G \times Z/2Z$.
(b) Show that the number of times that the trivial representation of $G$ occurs in $\rho$ equals the number of times that the trivial representation of $G \times Z/2Z$ occurs in $\bar{\rho}$. 