1. Let $G$ be a group of order $pq$, where $p$ and $q$ are primes with $p < q$.
   (a) Let $S$ be a Sylow $q$-subgroup of $G$ and let $g \in G$. Show that $g^p \in S$.
   (b) Let $G^p = \{g^p \mid g \in G\}$. Show that $G^p$ is a subgroup of $G$.

2. Let $M$ be an $n \times n$ matrix with entries in the complex numbers $\mathbb{C}$. Suppose $M$ has rank 1. Show that either $M^2 = 0$ or $M$ is diagonalizable.

3. Let $R$ be a commutative ring with 1 and suppose that all ideals of $R$ are projective when they are considered as modules over $R$. Let $M \subseteq R^n$ be a submodule of $R^n$ for some $n$. Show that $M$ is isomorphic to a direct sum of ideals of $R$. (Hint: Consider the projection of $M$ onto the last coordinate in $R^n$.)

4. (a) Let $R$ be a commutative ring with 1. Let $S$ be the set of ideals of $R$ that are not finitely generated. Use Zorn’s Lemma to show that if $S$ is not empty then $S$ contains a maximal element $I$ (when $S$ is ordered by inclusion).
   (b) Let $I$ be as in part (a) and let $\alpha \in R$ with $\alpha \notin I$. Show that the ideal generated by $I$ and $\alpha$ can be generated by $\alpha$ and finitely many elements $g_j \in I$.
   (c) Let $\alpha$ and $I$ be as above. Assume there exists $\beta \notin I$ such that $\alpha \beta \in I$. Let $J = \{r \in R \mid ra \in I\}$. Show that $J$ is finitely generated.
   (d) Assume $\alpha, \beta, I, J$ are as above. Let the generators of $J$ be $h_1, \ldots, h_m$. Show that $I$ is generated by the set of $g_j$’s and $\alpha h_1, \ldots, \alpha h_m$ (which contradicts the fact that $I$ is not finitely generated).
   (e) Show that if all the prime ideals of $R$ are finitely generated, then $R$ is Noetherian.

5. Let $L/K$ be a finite Galois extension of fields with $G = \text{Gal}(L/K)$. Let $\alpha \in L$. Let
   \[ f(X) = \prod_{g \in G} (X - g(\alpha)). \]
   (a) Show that $f(X) \in K[X]$.
   (b) Show that $f(X)$ is irreducible in $K[X]$ if and only if $\#(G\alpha) = \#G$, where “#” denotes cardinality and $G\alpha$ is the set of elements of the form $g(\alpha)$ with
6. Let $G$ be a finite group and let $R = \mathbb{C}[G]$ be the group ring of $G$ with complex coefficients. We may regard $R$ as a complex vector space of dimension equal to the order of $G$. Consider the representations $\rho_i : G \rightarrow GL(R)$, $i = 1, 2, 3$, defined by

\[
\begin{align*}
\rho_1(g)(r) &= grg^{-1}, \\
\rho_2(g)(r) &= gr, \\
\rho_3(g)(r) &= rg^{-1},
\end{align*}
\]

where $g \in G$ and $r \in R$.

(a) Show that the representations $\rho_2$ and $\rho_3$ are equivalent.

(b) Show that $\rho_1$ is equivalent to $\rho_2$ if and only if $G$ is trivial.