DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM
January 2013
ALGEBRA (Ph.D. Version)

Instructions to the student
a. Answer all six questions; each will be assigned a grade from 0 to 10.
b. Use a different booklet for each question. Write the problem number and your code number (not your name) on the outside of the booklet.
c. Keep scratch work on separate pages in the same booklet.

1. (a) Show that the group of automorphisms of \( \mathbb{Z}/221\mathbb{Z} \) has no element of order 5. (Note: 221 = 13 \times 17.)
(b) Show that a group of order 1105 = 5 \times 13 \times 17 must be cyclic.

2. Let \( A \in M_3(\mathbb{C}) \) be a 3 \times 3 matrix with minimal polynomial \( X^3 \). Let \( B \in M_4(\mathbb{C}) \) be a 4 \times 4 matrix with minimal polynomial \( X^4 \). The matrix \( A \otimes B \) acts on \( \mathbb{C}^3 \otimes \mathbb{C}^4 \cong \mathbb{C}^{12} \) by \((A \otimes B)(v \otimes w) = Av \otimes Bw\).
(a) Find the minimal polynomial of \( A \otimes B \) and justify your answer.
(b) Find the dimension of the kernel of \( A \otimes B \).

3. Let \( R \) be a principal ideal domain and let \( a, b \) be nonzero elements of \( R \) that are not units. The set of \( R \)-homomorphisms
\[
H = \text{Hom}_R \left( R \oplus R/aR, R \oplus R/bR \right)
\]
from \( R \oplus R/aR \) to \( R \oplus R/bR \) is an \( R \)-module, where the action of \( R \) is given by \((rf)(x) = r(f(x))\) (with \( r \in R \) and \( f \in H \)) and the addition is pointwise addition of functions. Describe the decomposition of \( H \) as a direct sum of cyclic \( R \)-modules. Justify your answer.

4. Let \( K \) be a field.
(a) Let \( L/K \) be an algebraic extension. Let \( R \) be a ring with \( K \subseteq R \subseteq L \). Show that \( R \) is a field.
(b) Suppose \( A \) is a commutative ring containing \( K \) and \( \phi : A \to K[X] \) is a ring homomorphism that is the identity on \( K \). Let \( M \) be a maximal ideal of \( K[X] \). Show that \( \phi^{-1}(M) \) is a maximal ideal of \( A \).

5. Let \( K \) be a field of characteristic 0 such that every odd degree polynomial \( f(X) \in K[X] \) has a root in \( K \).
(a) Let \( L/K \) be a finite Galois extension. Show that \([L : K]\) is a power of 2.
(b) Let \( L/K \) be a finite extension. Show that \([L : K]\) is a power of 2.

6. Let \( G \cong (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) \) be the group generated by \( a \) and \( b \) with relations \( a^4 = b^2 = 1 \) and \( ab = ba \). Let
\[
A = \begin{pmatrix} i & 0 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ 1 + i & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\]
(a) Show that there are matrices \( U \) and \( V \) such that \( UAU^{-1} = S \) and \( VBV^{-1} = T \).
(b) Show that the map \( \rho : G \to GL_2(\mathbb{C}) \) satisfying \( \rho(a^i b^j) = A^i B^j \) is a well-defined representation of \( G \).
(c) The homomorphism \( \pi : G \to GL_2(\mathbb{C}) \) satisfying \( \pi(a^i b^j) = S^i T^j \) is a well-defined representation of \( G \) (do not show this; the proof is similar to (b)). Show that \( \rho \) and \( \pi \) are not equivalent representations of \( G \).