1. Let $G$ be a group acting on a set $S$. Assume that the action is transitive (that is, given $s, t \in S$, there exists $g \in G$ such that $gs = t$), and that the action is faithful (that is, if $g \in G$ satisfies $gs = s$ for all $s \in S$, then $g = 1$). Fix $s \in S$ and let $G_s$ be the stabilizer of $s$. Show that if $N$ is a normal subgroup of $G$ such that $N \subseteq G_s$, then $N = 1$.

(b) Suppose $G$ is a finite abelian group that acts faithfully and transitively on a set $S$. Show that $G$ and $S$ have the same number of elements.

2. Let

$$M = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix},$$

and let $R = \mathbb{C}[M]$ be the span over the complex numbers of $I, M, M^2, M^3, \ldots$.

(a) Determine the dimension over $\mathbb{C}$ of $R$.

(b) Find two non-zero matrices $A$ and $B$ in $R$ such that $AB = 0$.

3. Let $p$ be prime, let $n \geq 1$. Let $R = (\mathbb{Z}/p^n\mathbb{Z})[X]/(X^p - 1)$.

(a) Let $M$ be the ideal of $R$ generated by $p$ and $X - 1$. Show that $M$ is a maximal ideal of $R$.

(Hint: Consider the homomorphism $f(X) \mapsto f(1) \mod p$.)

(b) Show that $M$ is the unique prime ideal of $R$. (Hint: Show that $(X - 1)^p \equiv 0 \pmod{p}$.)

4. Let $V$ and $W$ be finite-dimensional vector spaces over $\mathbb{C}$. Let $A$ be a linear transformation of $V$ and let $B$ be a linear transformation of $W$. Consider the map

$$A \otimes I - I \otimes B : V \otimes W \to V \otimes W$$

defined by mapping $v \otimes w$ to $(Av) \otimes w - v \otimes (Bw)$. Show that this map is invertible if and only if the set of eigenvalues of $A$ is disjoint from the set of eigenvalues of $B$.

5. Let $K$ be a field of characteristic 0. Let $n$ be a positive integer and assume that $K$ contains the $n$th roots of unity. Let $L/K$ be a finite Galois extension with Galois group $G$ and assume that $G$ is abelian of exponent $n$ (that is, $g^n = 1$ for all $g \in G$). Let

$$B = \{x \in K^\times \mid \sqrt[n]{x} \in L\}.$$ 

(a) Let $x \in B$ and $g \in G$. Choose $y \in L$ such that $y^n = x$. Show that

$$\langle x, g \rangle \overset{\text{def}}{=} \frac{g(y)}{y}.$$
is an $n$th root of unity that is independent of the choice of $y$ as an $n$th root of $x$.

(b) Show that if $g, h \in G$ and $x \in B$, then $(x, gh) = (x, g)(x, h)$.
(c) Let $x \in B$. Show that if $(x, g) = 1$ for all $g \in G$, then $x$ is the $n$th power of an element of $K^\times$.

(Note: You may not quote results from Kummer theory since they are what you are proving here.)

6. (a) Let $G$ be a finite group and let $A$ be a subgroup. Let $r_1, \ldots, r_d$ be a set of coset representatives for $G/A$. Let $\rho : G \rightarrow GL(V)$ be a finite-dimensional representation of $G$ over the complex numbers and let $W$ be a subspace of $V$ that is stable under the action of $A$ (that is, $\rho(A) \cdot W \subseteq W$). Let

$$U = \sum_{i=1}^{d} \rho(r_i) W.$$ 

Show that $U$ is stable under the action of $G$.

(b) Suppose that $(\rho, V)$ is an irreducible representation of a finite group $G$ and that $G$ has an abelian subgroup $A$. Show that $\dim(V) \leq [G : A]$. (Hint: What are the irreducible representations of $A$?)