1. Let $G$ be a group of order $168 = 3 \cdot 8 \cdot 7$, and assume that $G$ has no nontrivial normal subgroups.
   (a) Show that $G$ has eight Sylow 7-subgroups.
   (b) Let $H$ be a Sylow 7-subgroup of $G$ and let $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$. Show that $|N_G(H)| = 21$.
   (c) Show that $G$ has no subgroup of order 14. (Hint: There is no element of order 2 in $N_G(H)$.)

2. Let $A$ be an $n \times n$ matrix with entries in the complex numbers $\mathbb{C}$. Show that the following two statements are equivalent:
   i. $A$ is diagonalizable
   ii. For each $x \in \mathbb{C}$,
      \[\text{Image}(xI - A) + \text{Ker}(xI - A) = \mathbb{C}^n,\]
      where $I$ is the $n \times n$ identity matrix.

3. Let $R$ be a commutative ring with 1 and let $Q$ be a proper ideal of $R$. We say that $Q$ is primary if it satisfies the following condition: Let $x, y \in R$ with $xy \in Q$. If $x \notin Q$ then $y^n \in Q$ for some positive integer $n$ (depending on $y$). Define the radical of $Q$ by
   \[\text{rad}(Q) = \{x \in R \mid x^n \in Q \text{ for some positive integer } n\}.\]
   Note that rad($Q$) is also a proper ideal of $R$ (do not prove this).
   (a) Prove that if $Q$ is primary then rad($Q$) is a prime ideal.
   (b) Suppose $R$ is a principal ideal domain. Find all primary ideals of $R$ (prove that all primary ideals are on your list, and prove that all ideals on your list are primary).

4. Let $R$ be a commutative ring with 1 and let $M$ be a nonzero Noetherian $R$-module. Let $I = \{r \in R \mid rM = 0\}$.
   (a) Show that there is an injection $R/I \to M^n$ for some $n \geq 1$.
   (b) Show that $R/I$ is a Noetherian ring.

5. Let $L/K$ be a finite Galois extension of fields and let $\alpha \in L$. Suppose there is an automorphism $\sigma \in \text{Gal}(L/K)$ such that $\sigma(\alpha) = \alpha + 1$.
   (a) Show that $K$ has characteristic $p$ for some prime $p > 0$.
   (b) If $L = K(\alpha)$, show that $\sigma$ has order $p$. 

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(c) Suppose $\sigma$ generates $\text{Gal}(L/K)$. Show that $\alpha^p - \alpha \in K$.

6. Let $G$ be a finite group and let $\rho : G \to \text{GL}_n(\mathbb{C})$ be a representation of $G$. Let $\chi$ be the character of $\rho$.

(a) Define the matrix $M = \sum_{g \in G} \rho(g)$. Show that if $M \neq 0$ then there is a nonzero vector $v$ such that $\rho(g)v = v$ for all $g \in G$.

(b) Suppose $\sum_{g \in G} \chi(g) = 0$. Show that $M = 0$. 