Unless otherwise stated, you may appeal to a “well known theorem” in your solution to a problem, but if you do, it is your responsibility to make it clear which theorem you are using and why its use is justified. You may use any given hint without proving it. In problems with multiple parts, be sure to go on to the rest of the problem even if there is some part you cannot do. In working on any part, you may assume the answer to any previous part, even if you have not proved it.

1. Let $f$ be a real-valued everywhere differentiable function on $[0, 1]$. Prove that $f \in AC[0, 1]$ if and only if $f \in BV[0, 1]$.

2. Find a conformal map $h$ of the circular sector common to the circles $\{|z-1|=1\}$ and $\{|z-i|=1\}$ to the right half plane. Find $h^{-1}(1)$.

3. Let $f : [0, \infty) \to [0, \infty)$ be a uniformly continuous function so that $\int_0^\infty f(x)dx < \infty$.
   Show that $\lim_{x \to \infty} f(x)$ exists and $\lim_{x \to \infty} f(x) = 0$.
   Prove that $f$ uniformly continuous cannot be relaxed to $f$ being continuous, by constructing a continuous function $f : [0, \infty) \to [0, \infty)$ so that $\int_0^\infty f(x)dx < \infty$ but $f(x)$ does not converge to 0 as $x$ tends to $\infty$.

4. Let $f$ be an analytic homeomorphism from $\{0 < |z| < 1\}$ to $\{0 < |z| < 1\}$. Show that $f$ extends to an analytic homeomorphism from $\{|z| < 1\}$ to $\{|z| < 1\}$. Show that $f$ has the form $\alpha z$ for $|\alpha| = 1$.

5. (In this problem, you may not appeal to the Ergodic Theorem.) Let $m$ denote Lebesgue measure on $[0, 1]$. Suppose $T : [0, 1] \to [0, 1]$ is a Lebesgue measurable function such that for every measurable set $E \subset [0, 1]$, if $T^{-1}E = E$ then either $m(E) = 0$ or $m([0, 1] \setminus E) = 0$.
   (a) Suppose $f : [0, 1] \to \mathbb{R}$ is a Lebesgue measurable function such that $f(x) = f(T(x))$ for all $x$ in $[0, 1]$. Prove that there is a real number $c$ such that the set $\{x : f(x) = c\}$ has measure 1.
(b) Suppose also for every Lebesgue measurable subset $E$ of $[0,1]$ with $m(E) > 0$, there is a subset $F$ of $[0,1]$ with $m([0,1] \setminus F) = 0$ such that for all $x \in F$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_E(T^k(x)) = m(E).$$

Given $f : [0,1] \to \mathbb{R}$ is bounded and measurable, prove that the sequence $\{f_n\}_{n \geq 1}$ defined by

$$f_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x))$$

converges to $\int_0^1 f \, dm$ in $L^1([0,1], m)$.

6. Let $\mathcal{U}$ be the set of real-valued functions that: on $\{|z| < 1\}$ are harmonic and on $\{|z| \leq 1\}$ are continuous and have absolute value bounded by a positive constant $M$. Show that the $k^{th}$, $k > 0$, derivatives of elements of $\mathcal{U}$ are uniformly bounded on compact subsets of $\{|z| < 1\}$. Explain why $\mathcal{U}$ is a normal family of harmonic functions.