ANALYSIS QUALIFYING
EXAMINATION

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Unless otherwise stated, you may appeal to a "well known theorem" in your solution to a problem, but if you do, it is your responsibility to make it clear which theorem you are using and why its use is justified. You may use any given hint without proving it. In problems with multiple parts, be sure to go on to the rest of the problem even if there is some part you cannot do. In working on any part, you may assume the answer to any previous part, even if you have not proved it.

1. Let \( \{A_n\}_{n \geq 1} \) be a sequence of Lebesgue measurable subsets of \([0, 1]\). Assume that 1 is a limit point of the sequence \( \{m(A_n)\}_{n \geq 1} \), where \( m \) denotes the Lebesgue measure on \([0, 1]\). Prove that there exists a subsequence \( A_{n_k} \) such that

\[
m(\bigcap_{k=1}^{\infty} A_{n_k}) > 0.
\]

2. Evaluate the integral

\[
\int_{0}^{\infty} \frac{dx}{x^4 + 1}.
\]

3. Let \( f, f' \in L^1(\mathbb{R}) \) and assume that \( f \) is absolutely continuous on \( I \) for each bounded interval \( I \) in \( \mathbb{R} \). Prove that

\[
\int_{-\infty}^{\infty} f'(x)dx = 0.
\]

4. Without using Picard's theorems, show that if \( f(z) \) is analytic and one-to-one on \( \mathbb{C} \), then \( f(z) \) has the form \( az + b \) for appropriate constants.
5. Given $a < b$, denote by $m$ the Lebesgue measure on $[a, b]$. Let $f$ be a positive (i.e., $f > 0$ on $[a, b]$) and Lebesgue measurable function defined on $[a, b]$ such that
\[ f \in L^r([a, b], dm) = \{ f : [a, b] \to (0, \infty) : \int_a^b f(x)^r dx < \infty \} \]
for some $r > 0$.

(a) Show that $f \in L^s([a, b], dm)$ for each $0 < s \leq r$.
(b) Show that $\lim_{s \to 0^+} \int_a^b f(x)^s dx = b - a$ and conclude that
\[ \lim_{s \to 0^+} \left( \int_a^b f(x)^s dx \right)^{1/s} = \infty \text{ if } b - a > 1 \]
and
\[ \lim_{s \to 0^+} \left( \int_a^b f(x)^s dx \right)^{1/s} = 0 \text{ if } b - a < 1. \]

6. For a complex parameter $\lambda$, $|\lambda| < 2$, consider the solutions to the equation
\[ z^4 - 4z + \lambda = 0 \]  \hspace{1cm} (1)
(a) Show that there is exactly one solution $z(\lambda)$ to equation (1) with $|z(\lambda)| < 1$
(b) Show that the function $\lambda \to z(\lambda)$ is analytic for $|\lambda| < 2$.  
(c) What is the order of vanishing of $z(\lambda)$ at $\lambda = 0$?