1. Let $m^*$ be the Lebesgue outer measure on $\mathbb{R}$. Assume that $A, B \subset \mathbb{R}$ are such that there exists $\epsilon > 0$ with

$$\inf\{|a - b| : a \in A, b \in B\} \geq \epsilon.$$ 

Prove that $m^*(A \cup B) = m^*(A) + m^*(B)$.

2. Let $D$ be the unit disc $\{|z| < 1\}$ and $f : D \to \mathbb{C}$ a continuous function. Assume $f$ is analytic in $D_+ = \{z \in D \mid \Re z > 0\}$ and also is analytic in $D_- = \{z \in D \mid \Re z < 0\}$. Prove that $f$ is analytic in the disc $D$.

3. Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of Lebesgue measurable functions on $\mathbb{R}$ such that for some $a > 1$,

$$\int_{\mathbb{R}} |f_n| \, dm \leq \frac{1}{n^a}, n = 1, 2, 3, \ldots.$$ 

Prove that $\lim_{n \to \infty} f_n = 0$ a.e., where $m$ denotes the Lebesgue measure on $\mathbb{R}$. 
4. Let $\Omega \subset \mathbb{C}$ be open and assume $z_1$ and $z_2$ belong to the same connected component of the complement of $\Omega$.
   
   (a) Prove there exists a holomorphic function $f$ on $\Omega$ such that $f'(z) = \frac{1}{z-z_1} - \frac{1}{z-z_2}$ for all $z \in \Omega$.
   
   (b) Prove there exists a holomorphic function $g$ on $\Omega$ such that $e^{g(z)} = \frac{z-z_1}{z-z_2}$.

5. Assume that $f$ is absolutely continuous on $[0, a]$ for every $a \geq 0$, and $f(0) = 0$. Prove that
   
   $$\int_0^x |f(t)f'(t)|dt \leq \frac{1}{2} \left( \int_0^x |f'(t)|^2 dt \right)^2$$
   
   for all $x \geq 0$.

   Justify all your calculations.

6. Suppose that $f(z)$ is analytic on the unit disc $D = \{ |z| < 1 \}$ with $f(0) = 0$ and $f(D) \subset D$. Define the sequence of $n^{th}$ iterates by
   
   $$f^n = f \circ f \circ \cdots \circ f.$$

   Show that $f^n \to 0$ uniformly on compact subsets of $D$, except for the special case $f(z) = e^{i\theta_0}z$.
   
   (First consider cases for the maximum of $|f|$ on a proper subdisc of $D$.)