Problem solving in Complex Analysis is based on following model arguments. A model argument is a fixed outline of steps, and details are filled in for the specific problem.

For working problems with Schwarz’s Lemma: divide by the comparison function, use the given boundary bound and Maximum Principle to conclude that the quotient is bounded, then restate the bound for the original function. The comparison function is selected to match the geometry of the domain of the function.

For problems with isolated singularities: either the function is bounded, or its absolute value tends to infinity or its behavior is governed by the Casorati-Weierstrass Theorem (function approximately has every value in every neighborhood).

For problems where a bound is given for a holomorphic function on/near the boundary of a region then to estimate the function in the interior the possibilities are: Schwarz Lemma, Cauchy Integral formula (also for the derivatives) and the Maximum Principle.

Cauchy’s Theorem, the Cauchy Integral Formula, Cauchy’s estimate, Local Mapping Principle, the Argument Principle and Residue Theorem all have model arguments for working problems. Many proofs of theorems are even applications of model arguments and so provide models for solving problems.

Graduate Complex Analysis questions commonly involve only a single model argument. The first step in parsing a question is to match up the given information with the information/hypotheses for a single model argument.

Graduate Complex analysis includes the following topics.

**Basics**

Vector calculus, topology of sets in $\mathbb{C}$ (compact, connected, simply connected, Jordan Curve Theorem), trigonometric form $re^{i \theta}$, powers & roots of complex numbers, logarithm, definition of arg and its principal branch, $\sin \pi z$ and $\tan \pi z$, uniform and absolute convergence of power series, including the radius of convergence.

**Holomorphic functions**

The complex derivative $f'(z)$, Cauchy-Riemann equations, zeros, poles and Cauchy’s Theorem. Partial fraction decomposition. Geometry of Möbius transformations (mapping circle/lines to circle lines; symmetry about a line/circle; determined by three points.)

**Cauchy Integral Formula**

Winding number, Cauchy Integral Formula for $f^{(n)}(z)$, Cauchy’s estimate $|f^{(n)}(a)| \leq M n! r^{-n}$; Morera’s Theorem, Liouville’s Theorem, Residue Theorem (evaluations for rational functions, and with branches of $\log z$ and $z^\alpha$.)

**Local properties**

Isolated singularities, removable conditions, Casorati-Weierstrass Theorem on essential singularities, open mappings, Local Representations when $f(z_0) = w_0$: $f(z) - w_0 = (z - z_0)^n k(z) =$
\[ g((z - z_0)^n) = (h(z - z_0))^n \text{ with } k(z_0) \neq 0, \ g'(0) \neq 0, \ h'(0) \neq 0. \]

**Maximum Principle and Schwarz’s Lemma**

Facility with comparison functions for simple domains.

**Argument/Winding Principle**

The range of a holomorphic function contains those values for which (the image of a closed curve in the domain by) the function winds around the value. Rouché’s Theorem. Fundamental Theorem of Algebra.

**Harmonic functions**

Definition, Mean Value Property, harmonic conjugate, Poisson Integral Formula, Schwarz Reflection.

**Series of functions**

Taylor and Laurent series, uniform and absolute convergence, and infinite sums. Infinite products, absolute convergence for \( \prod_n (1 + a_n) \) and the Mittag-Leffler Theorem. (Not on the Qualifying Exam: \( \frac{x^2}{\sin^2 \pi x} \), and the Riemann zeta \( \zeta(s) \).)

**Normal Families**

Equicontinuous and normal families, the Riemann Mapping Theorem. There are TWO notions of normal family: normal in \( \mathbb{C} \) - converges to a holomorphic function (a little beyond syllabus: a family of holomorphic functions omitting two finite values is normal is \( \mathbb{C} \) normal) AND normal in \( \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \) - converges to a meromorphic function (a little beyond syllabus: a family of meromorphic functions omitting three values in \( \hat{\mathbb{C}} \) is \( \hat{\mathbb{C}} \) normal).

**References**


