1. (a) Let $L$ be the language whose only non-logical symbol is the binary relation symbol $E$. Let $K$ be the class of all $L$-structures $\mathfrak{A}$ such that $E^\mathfrak{A}$ is an equivalence relation with at least one finite equivalence class. Prove that there is no $L$-theory $T$ such that $K = \text{Mod}(T)$.

**Hint:** Assuming that $K \subseteq \text{Mod}(T)$ show that $T$ has a model which is not in $K$.

(b) Let $L$ be a language containing at least the binary relation symbol $E$. Let $T$ be a theory of $L$ such that $E$ defines an equivalence relation in each model of $T$. Assume that $T$ has some model which has arbitrarily large finite $E$-classes but no infinite $E$-class. Prove that $T$ has a model with infinitely many infinite $E$-classes.

2. (a) Let $T$ be a complete theory in a countable language $L$. Assume there is a type $\Phi(x)$ consistent with $T$ such that any two countable models of $T$ realizing $\Phi$ are isomorphic. Prove that $T$ has a countable $\omega$-saturated model.

(b) Let $\mathfrak{A}_0$, $\mathfrak{B}$, and $\mathfrak{A}_1$ be $L$-structures. Assume that $\mathfrak{A}_0 \prec \mathfrak{A}_1$ and $\mathfrak{A}_0 \subseteq \mathfrak{B} \subseteq \mathfrak{A}_1$. Prove that if $\mathfrak{A}_0 \models \sigma$ then $\mathfrak{B} \models \sigma$ for every $L$-sentence $\sigma$ of the form

$$\exists x_1 \ldots \exists x_n \forall y_1 \ldots \forall y_m \alpha(x_1, \ldots, x_n, y_1, \ldots, y_m),$$

where $\alpha$ is an open $L$-formula.

3. (a) Let $T$ be a complete theory in a countable language $L$, and assume that $T$ has a prime model $\mathfrak{A}$. Assume that no (countable) proper elementary extension of $\mathfrak{A}$ is also prime. Prove that no uncountable model of $T$ is atomic.

(b) Let $T$ be a complete theory in a language $L$, and let $\Phi(x)$ be an $L$-type. Assume that $\Phi$ is realized by at most two elements in every model of
$T$ and by exactly two elements in some model of $T$. Prove that $\Phi$ is realized by exactly two elements in every model of $T$.

4. (a) Prove that there is some $H : \omega \to \omega$ such that for every recursive $f : \omega \to \omega$ there is some $n_f \in \omega$ such that $f(k) < H(k)$ for all $k > n_f$.

(b) Let $A \subseteq \omega$ be an infinite r.e. set. Prove that there are disjoint infinite r.e. sets $A_0$ and $A_1$ with $A = A_0 \cup A_1$.

5. (a) Let $L$ be the language whose only non-logical symbol is the binary relation symbol $E$. Prove that there is an undecidable theory $T$ of $L$ such that $E^\mathfrak{a}$ is an equivalence relation in every $\mathfrak{a} \models T$.

(b) Prove that there is some $e \in \omega$ such that

$$\{e\}(e + 1) = (e + 2)(e).$$

**Note:** You may not use the special properties of any particular enumeration of the partial recursive functions.

6. (a) Let $f, g : \omega \to \omega$ be total recursive functions. Let $I_f = \{e : \{e\} = f\}$ and let $I_g = \{e : \{e\} = g\}$. Prove that $I_f \equiv_m I_g$.

(b) Let $f : \omega \to \omega$ be a total recursive function and let $I_f = \{e : \{e\} = f\}$. Prove that $I_f \leq_T \emptyset^\prime$. 

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