1. (a) Let $T$ be any $L$-theory and suppose that $\{\varphi_n(x) : n \in \omega\}$ are $L$-formulas such that $T \models \forall x (\varphi_n(x) \rightarrow \varphi_{n+1}(x))$ for all $n \in \omega$. Suppose further that every element of every model of $T$ realizes some $\varphi_n$. Prove that $T \models \forall x \varphi_n(x)$ for some $n \in \omega$.

(b) Let $\mathfrak{A}$ be an $L$-structure, let $a \in A$, and assume that $a$ satisfies some complete $L$-formula in $\mathfrak{A}$. Let $L' = L \cup \{c\}$, and let $\mathfrak{A}'$ be the expansion of $\mathfrak{A}$ to an $L'$-structure in which $c^{\mathfrak{A}'} = a$. Suppose that $b \in A$ and that $b$ satisfies a complete $L'$-formula in $\mathfrak{A}'$. Prove that the pair $ab$ satisfies a complete $L$-formula in $\mathfrak{A}$.

2. A theory $T$ is called model complete if every embedding of models of $T$ is an elementary embedding.

(a) Suppose that $L = \{E\}$ and $T$ is the $L$-theory asserting that $E$ is an equivalence relation with infinitely many classes, and each class is infinite. Prove that $T$ is model complete.

(b) Prove that if $T$ is model complete, then for every $L$-formula $\varphi(x_1, \ldots, x_n)$, there is an existential $L$-formula $\psi(x_1, \ldots, x_n)$ such that

$$T \models \forall x (\varphi(x) \leftrightarrow \psi(x))$$

3. (a) Suppose $L = \{U, \le\}$, where $U$ is a unary predicate and $\le$ is binary. Let $\mathfrak{A}$ be the $L$-structure with universe $\mathbb{R}$ (the real numbers), where $U^\mathfrak{A} = \mathbb{Q}$ (the rationals) and $\le^\mathfrak{A}$ is the usual ordering on $\mathbb{R}$. Find, with proof, all countable models of $Th(\mathfrak{A})$, up to isomorphism.

(b) Prove that if $T$ is $\omega$-categorical and $\mathfrak{A}$ is the infinite, countable model, then there is $\mathfrak{B} \cong \mathfrak{A}$ with $\mathfrak{B} \neq \mathfrak{A}$. 
4. (a) Prove that $Th(\mathfrak{M})$, where $\mathfrak{M} = (\omega, +, \cdot, 0, s)$, is not model complete (see Problem #2).

(b) Assume that $PA + \text{Con}(PA)$ is consistent. Use Gödel’s Second Incompleteness Theorem to conclude that $PA + \neg\text{Con}(PA)$ is consistent.

5. (a) Prove that there is an integer $m$ so that $W_m = \{m\}$.

(b) Let $Z = \{e : W_e \neq \emptyset\}$. Prove that $Z$ is a many-one complete, recursively enumerable subset of $\omega$.

6. (a) Determine (with proof) whether or not $\text{TOT} = \{e : \text{e is total}\}$ is Turing equivalent to $\text{FIN} = \{e : W_e \text{ is finite}\}$.

(b) Demonstrate that $\{e : W_e \text{ is recursive}\}$ is an arithmetic subset of $\omega$. 