1. (a) Suppose $T$ is a theory in a language with only finitely many non-logical symbols. Prove that if $T$ has infinitely many non-isomorphic models, then $T$ has an infinite model.

(b) Suppose $L \subseteq L'$ are languages, $\mathfrak{A}$ is an $L$-structure, and $T'$ is a consistent $L'$-theory. Additionally, assume that there is no model of $T'$ whose reduct to $L$ is elementarily equivalent to $\mathfrak{A}$. Prove that there is an $L$-sentence $\theta$ such that $\mathfrak{A} \models \theta$, but $T' \models \neg \theta$.

2. (a) Let $L = \{E\}$, where $E$ is a binary relation, and let $T$ be the $L$-theory asserting that $E$ is an equivalence relation with infinitely many classes, and that each class is infinite. Prove that $T$ is model complete, i.e., for all models $\mathfrak{A}, \mathfrak{B} \models T$, $\mathfrak{A} \subseteq \mathfrak{B}$ implies $\mathfrak{A} \preceq \mathfrak{B}$.

(b) Let $\mathfrak{A}$ be any proper elementary extension of $\mathfrak{N} = (\omega, +, \cdot, <)$. An initial substructure is a substructure (not necessarily elementary) $\mathfrak{B} \subseteq \mathfrak{A}$ in which the set $B$ is a $<$-initial segment of $A$. Prove that for any $a \in A$ there is an initial substructure $\mathfrak{B} \subseteq \mathfrak{A}$ with $a \in B$, but $B \neq A$. [Possible hint: Recall that there is an $L$-formula $\varphi(x, y, z)$ such that $k^\ell = m$ if and only if $\mathfrak{N} \models \varphi(k, \ell, m)$ for all $k, \ell, m \in \omega$.]

3. Suppose that $T$ is a complete theory in a countable language.

(a) Prove directly from the definitions that if $\mathfrak{A} \models T$ is countable and atomic, then it embeds elementarily into every model of $T$. It is not sufficient to simply quote theorems from class.

(b) Suppose that some atomic $\mathfrak{A} \models T$ has a proper, elementary substructure. Prove that $T$ has an uncountable, atomic model.
4. (a) Assume that $R \subseteq \omega^2$ is recursively enumerable and that the sets 
\{R_k : k \in \omega\} are all infinite and are pairwise disjoint. Prove that 
there is a recursive set $C \subseteq \omega$ that intersects each $R_k$ in exactly 
one point.

(b) Prove that every decidable theory in a language with finitely many 
non-logical symbols has a complete, decidable extension.

5. Let $Fm_x$ denote the set of formulas in the language $L = \{+, \cdot, <, s, 0\}$ 
whose free variables is precisely $\{x\}$. For each $\varphi(x) \in Fm_x$, let $d\varphi$ 
denote the sentence $\exists x(x = \lceil \varphi \rceil \land \varphi(x))$. Let $f : \omega \rightarrow \omega$ be the 
(recursive) function 
\[ f(n) = \begin{cases} \lceil d\varphi \rceil & \text{if } n = \lceil \varphi \rceil \text{ for some } \varphi \in Fm_x \\ 0 & \text{otherwise} \end{cases} \]
and let $T$ be any theory in which $f$ is represented.

(a) Prove that for every formula $\theta(x) \in Fm_x$ there is a sentence $\psi$ 
such that $T \vdash \psi \leftrightarrow \theta(\lceil \psi \rceil)$.

(b) Prove that if $T$ is a consistent theory in which every recursive 
function is represented, then $T$ is undecidable.

6. (a) Prove that $\{k \in \omega : \varphi_{2k}(3k) \uparrow\}$ is $\Pi_1$ but not $\Delta_1$.

(b) Prove that $\text{INF}$ is many-one reducible to $\text{ZERO}$, where $\text{INF} = \{e \in \omega : W_e \text{ is infinite}\}$ and $\text{ZERO} = \{e \in \omega : \forall n \varphi_e(n) = 0\}$. 
1. (a) Prove that the class of cyclic groups is not an elementary class. (Recall that a group $G$ is cyclic iff there is some $g \in G$ such that $G = \{g^n : n \in \mathbb{Z}\}$.)

(b) Prove that every countable linear order embeds isomorphically into $(\mathbb{Q}, \leq)$.

2. (a) Let $L_1 = \{U\}$, where $U$ is a unary predicate symbol. Prove that for any $L_1$-sentence $\theta$, if $\theta$ is true in every finite $L_1$-structure, then $\theta$ is valid.

(b) Let $L_2 = \{R\}$, where $R$ is a binary predicate symbol. Find (with proof) an $L_2$-sentence $\theta$ such that $\theta$ holds in every finite $L_2$-structure, but $\theta$ is not valid.

3. (a) Prove that no complete theory $T$ extending Peano's Axioms can have a countable, saturated model.

(b) Let $T$ be a complete theory in a countable language, and let $\Gamma(x)$, $\Phi(x)$ be 1-types such that (1) there is a model of $T$ omitting $\Gamma$ and (2) every model of $T$ that omits $\Gamma$ realizes $\Phi$. Prove that $\Phi$ is realized in every model of $T$. 

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4. (a) Prove that there is a model $\mathfrak{A}$ of Peano’s Axioms and a formula $\theta(x)$ such that $\mathfrak{A} \models \exists x \theta(x)$, yet $\mathfrak{A} \not\models \theta(n)$ for every $n \in \omega$.

(b) Suppose $L$ has only finitely many nonlogical symbols, and $T$ is a finitely axiomatizable $L$-theory such that for any $L$-sentence $\theta$, if $\theta$ is not true in every model of $T$, then $\theta$ is false in some finite model of $T$. Prove that $T$ is decidable.

5. (a) Prove that there is no total recursive $f : \omega \rightarrow \omega$ such that for all $e \in \omega$, if $W_e$ is finite, then $W_e \subseteq \{0, 1, \ldots, f(e)\}$.

(b) Construct an r.e. subset $A \subseteq \omega$ such that $\omega \setminus A$ is infinite, but $A \cap B$ is nonempty for every infinite, r.e. set $B$.

6. (a) Give an example (with justifications) of two sets $A, B \subseteq \omega$ such that $A$ is Turing reducible to $B$, but $A$ is not many-one reducible to $B$.

(b) Exhibit (with proof) two disjoint, r.e. sets $A$ and $B$ that are recursively inseparable, i.e., there is no recursive $C$ such that $A \subseteq C$, but $B \cap C = \emptyset$. 
1. Suppose that $L \subseteq L'$ are languages, $\mathcal{A}$ is an $L$-structure, and $T'$ is an $L'$-theory such that $T' \cup Th_L(\mathcal{A})$ is consistent.
   
   (a) Prove that there is an $L'$-structure $\mathcal{B} \models T'$ such that the $L$-reduct, $\mathcal{B} = \mathcal{B}'|_L$ elementarily extends $\mathcal{A}$.
   
   (b) Prove that there is a model of $T'$ realizing every 1-type $\Gamma(x)$ in the language $L$ consistent with $Th(\mathcal{A})$.

2. Let $D(x, y)$ denote the divisibility relation on $\omega$, i.e., $D(n, m)$ if and only if $n$ divides $m$. Let $\mathcal{A} = (\omega, D)$.
   
   (a) Prove that the set of primes is definable in $\mathcal{A}$.
   
   (b) Prove that $\mathcal{A}$ has a nontrivial automorphism, i.e., an isomorphism $f : \mathcal{A} \rightarrow \mathcal{A}$ such that $f(n) \neq n$ for at least one $n \in \omega$.

3. (a) Prove that if $\mathcal{A}$ is an infinite, countable, saturated model then there is a countable, saturated $\mathcal{B} \subseteq \mathcal{A}$ with $\mathcal{B} \neq \mathcal{A}$.
   
   (b) Let $\mathcal{A}_0 \preceq \mathcal{B}_0 \preceq \mathcal{A}_1 \preceq \mathcal{B}_1 \preceq \mathcal{A}_2 \preceq \ldots$ be an elementary chain of models where each $\mathcal{A}_n$ is countable and saturated, and each $\mathcal{B}_n$ is not saturated. Prove that $\bigcup_{n \in \omega} \mathcal{B}_n$ is countable and saturated.
4. (a) Let $\mathcal{N} = (\omega, +, \cdot, 0, 1)$ denote the standard model of arithmetic, and let $PA$ denote Peano's axioms. Prove that there is a countable $\mathfrak{A} \models PA$ such that $\mathfrak{N} \subseteq \mathfrak{A}$, but $\mathfrak{N} \not\subseteq \mathfrak{A}$.

(b) Given a binary function $g : \omega \times \omega \rightarrow \omega$, let $g^*$ be the partial function defined by

$$g^*(x) = \begin{cases} y & \text{if, for some } n, g(m, x) = y \text{ for all } m \geq n \\ \uparrow & \text{otherwise} \end{cases}$$

Construct a (total) recursive $g : \omega \times \omega \rightarrow \omega$ such that the domain of $g^*$ is a non-recursively enumerable set, e.g., $\overline{K}$.

5. Let $E(x, y) = x^y$ denote the exponential function.

(a) Prove that the graph of multiplication is definable in the structure $(\omega, E)$.

(b) Prove that the structure $(\omega, E)$ is strongly undecidable.

6. For $X \subseteq \omega$, let $S_X = \{ e \in \omega : W_e = X \}$

(a) Prove that $S_X$ is $\Pi_3$ for every recursive set $X$.

(b) Find (with proof) a recursive $X \subseteq \omega$ such that $S_X$ is not $\Pi_3$-complete.
1. (a) Let $\mathfrak{A}$ and $\mathfrak{B}$ be elementarily equivalent structures in the same language $L$. Prove that there is an $L$-structure $\mathfrak{C}$ and elementary embeddings $f : \mathfrak{A} \rightarrow \mathfrak{C}$ and $g : \mathfrak{B} \rightarrow \mathfrak{C}$.

(b) Let $L = \{<, U\}$, where $U$ is unary and $<$ is binary. Let $\mathfrak{A}$ be any $L$-structure with universe the rationals $\mathbb{Q}$, where $<^\mathfrak{A}$ is interpreted as the usual ordering on $\mathbb{Q}$ and $U^\mathfrak{A}$ is any dense, codense subset, e.g.,

$$U^\mathfrak{A} = \left\{ \frac{n}{2k} : n, k \text{ are integers} \right\}$$

Prove that $Th(\mathfrak{A})$ is $\omega$-categorical.

2. (a) Let $L = \{+, \cdot, 0, 1\}$ and let $\mathfrak{N} = (\omega, +, \cdot, 0, 1)$ be the standard model of arithmetic. Let $\varphi(x)$ be any $L$-formula defining the set of prime numbers in $\omega$. Prove that if $\mathfrak{A}$ is an elementary extension of $\mathfrak{N}$ and $\mathfrak{A} \neq \mathfrak{N}$, then there is $a \in A \setminus \omega$ such that $\mathfrak{A} \models \varphi(a)$.

(b) Prove that every model (even the uncountable ones) of an $\omega$-categorical theory in a countable language is atomic.

3. Let $T$ be a complete theory in a countable language.

(a) Prove that if $\mathfrak{A}$ is a countably universal model of $T$, then $\mathfrak{A}$ has an $\omega$-saturated elementary substructure.

(b) Prove that if $\mathfrak{A}$ is an infinite, countable, $\omega$-saturated model of $T$, then $\mathfrak{A}$ has a nontrivial automorphism, i.e., an isomorphism $f : \mathfrak{A} \rightarrow \mathfrak{A}$ such that $f(a) \neq a$ for at least one $a \in A$. 


4. Let $L = \{ f \}$, where $f$ is a binary function symbol, and let $Valid_L$ denote the set of valid sentences in this language.

(a) Prove that $Valid_L$ is not essentially undecidable.

(b) Find an $L$-sentence $\sigma \notin Valid_L$, yet $\sigma$ holds in every finite $L$-structure.

5. (a) Suppose that every recursively enumerable set $A$ is many-one reducible to a fixed set $B \subseteq \omega$. Prove that $B$ contains an infinite, recursively enumerable subset.

(b) Let $A = \{ e \in \omega : W_e \text{ is finite} \}$ and $B = \{ e \in \omega : W_e \text{ is infinite} \}$. Prove that $A$ is Turing reducible to $B$, but not many-one reducible to $B$.

6. (a) Prove or disprove: If a binary relation $R$ is r.e. and $|R_k| \leq 2$ for each $k$, then $R$ is recursive.

(b) Let $A \subseteq \omega$ be weakly represented, but not represented by a formula $\varphi(x)$ with respect to $Q$. Prove that there is a consistent, recursively axiomatizable theory $T \supseteq Q$ such that $A$ is not weakly represented by $\varphi(x)$ with respect to $T$. 
1. (a) Prove that if $\mathfrak{A} \preceq \mathfrak{B}$ and $A$ is finite, then $\mathfrak{A} = \mathfrak{B}$.

(b) Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are structures in the same language $L$ that satisfy the same universal sentences. Prove that there is an $L$-structure $\mathfrak{C}$ into which both $\mathfrak{A}$ and $\mathfrak{B}$ embed isomorphically.

2. (a) Find (with proof) all automorphisms of the structure $\mathfrak{A} = (\mathbb{Z}, +)$.

(b) Recall that a countable $\mathfrak{A} \models T$ is $\omega$-homogeneous iff for all $n \in \omega$ and all $a_0, \ldots, a_n, b_0, \ldots, b_n \in A$ there is an automorphism $h$ of $\mathfrak{A}$ such that $h(a_i) = b_i$ for all $0 \leq i \leq n$ whenever $tp_{\mathfrak{A}}(a_0, \ldots, a_n) = tp_{\mathfrak{A}}(b_0, \ldots, b_n)$.

Prove that if $\mathfrak{A}$ and $\mathfrak{B}$ are both countable, $\omega$-homogeneous models of $T$, $\mathfrak{A}$ embeds elementarily into $\mathfrak{B}$, and $\mathfrak{B}$ embeds elementarily into $\mathfrak{A}$, then $\mathfrak{A} \cong \mathfrak{B}$.

3. Let $T$ be a complete theory in a countable language.

(a) Prove that if $T$ does not have a prime model, then $T$ has uncountably many nonisomorphic countable models.

(b) Let $X$ be a countable set of 1-types such that for every finite $F \subseteq X$ there is a model $\mathfrak{A}_F \models T$ omitting every $\Phi \in F$. Prove that there is a model $\mathfrak{B} \models T$ omitting every $\Phi \in X$. 
4. (a) Suppose that \( T \) is a recursively axiomatizable theory in a finite language \( L \) that has no infinite models. Prove that \( T \) is decidable.

(b) Let \( L = \{+,-,0,s,<\} \) and let \( Valid_L \) denote the set of valid \( L \)-sentences. Prove that \( Valid_L \) is undecidable, but not essentially undecidable.

5. (a) Let \( T \) be any consistent, recursively axiomatizable extension of Robinson’s \( Q \) and let \( Thm_T = \{ \sigma \in T : T \vdash \sigma \} \). Prove that \( Thm_T \) is weakly represented in \( Q \), but is not represented in \( Q \).

(b) Let \( PA \) denote Peano’s Axioms. Use Gödel’s 2nd Incompleteness Theorem to prove that if \( PA \) is consistent, then

\[
PA \cup \{Con(PA + \neg Con(PA))\}
\]

has a model.

6. Let \( K = \{ e \in \omega : \{e\}(e) \downarrow \} \) and \( Even = \{ e \in \omega : W_e = \{2n : n \in \omega \} \} \).

(a) Prove that there is an infinite, r.e. \( B \) such that \( K \) and \( B \) are recursively inseparable.

(b) Prove that \( Even \leq_r 0’ \).
1. (a) Let $T$ be any theory in a language $L$ that has an infinite model. Prove that $T$ has a model $\mathfrak{A}$ with an element $a \in A$ such that $a \neq c^\mathfrak{A}$ for every constant symbol $c \in L$.

(b) Suppose that $\mathfrak{A}$ is a saturated model of $Th(\mathfrak{A})$, and that a complete 1-type $\Phi(x)$ is realized by only finitely many elements of $\mathfrak{A}$. Prove that there is a formula $\varphi(x) \in \Phi(x)$ such that $\varphi$ is realized by only finitely many elements of $\mathfrak{A}$.

2. (a) Let $L^\text{nil} = \{+, \cdot, 0, 1, \leq\}$. Prove that any proper elementary extension $\mathfrak{B} > (\mathbb{R}, +, \cdot, 0, 1, \leq)$ contains an element $b \in B$ such that $\mathfrak{B} \models b > r$ for every $r \in \mathbb{R}$.

(b) Recall that a countable model $\mathfrak{A}$ is $\omega$-homogeneous iff for all $n \in \omega$ and all $a_0, \ldots, a_n, b_0, \ldots, b_n \in A$ there is an automorphism $h$ of $\mathfrak{A}$ such that $h(a_i) = b_i$ for all $0 \leq i \leq n$ whenever $tp_\mathfrak{A}(a_0, \ldots, a_n) = tp_\mathfrak{A}(b_0, \ldots, b_n)$.

Prove that every countable model in a countable language has a countable, $\omega$-homogeneous elementary extension.

3. Let $L^\text{nil} = \{E\}$, where $E$ is a binary relation symbol. Let $T$ be the theory asserting that $E$ is an equivalence relation with exactly two classes, both of which are infinite.

(a) Prove that $T$ is a complete $L$-theory.

(b) Prove that if $\mathfrak{A}$ and $\mathfrak{B}$ are models of $T$ and $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{A} \prec \mathfrak{B}$.
4. (a) Suppose that $T$ is a recursively axiomatizable theory with a model $\mathfrak{A} \models T$ that embeds elementarily into every model of $T$. Prove that $T$ is decidable.

(b) Assume that $A \subseteq \omega$ is recursive, $R \subseteq \omega \times \omega$ is r.e., and that $\bigcup_{k \in \omega} R_k = A$. Prove that there is a recursive $S \subseteq R$ such that $\bigcup_{k \in \omega} S_k = A$.

5. Let $\mathcal{F} = \{\text{all functions } f : \omega \to \omega \text{ such that } f(n+1) = nf(n) \text{ for all but finitely many } n \in \omega\}$.

(a) Prove that every $f \in \mathcal{F}$ is recursive.

(b) Prove that there is a recursive function $g : \omega \to \omega$ such that for every $f \in \mathcal{F}$ there is an $N \in \omega$ such that $g(n) \geq f(n)$ for every $n \geq N$.

6. (a) Let $T$ be a consistent, recursively axiomatizable theory containing the axioms for $Q$. Prove that for every formula $\varphi(x)$ of the language for $Q$ there is a sentence $\sigma$ such that $T \vdash \sigma \iff \varphi(\overline{x})$.

(b) Recall that $K = \{e : \{e\}(e) \downarrow\}$ and $\overline{K} = \omega \setminus K$. Prove that $K$ is not many-one reducible to $\overline{K}$. 
1. a) Prove or disprove: \((\mathbb{Z}, <)\) has a proper elementary substructure.

b) Let \(L^{nl} = \{E\}\) where \(E\) is a binary relation symbol. Let \(\mathfrak{A}\) be the countable \(L\)-structure in which \(E^\mathfrak{A}\) is an equivalence relation such that \(E^\mathfrak{A}\) has no infinite equivalence classes and for every \(n \geq 1\) there is exactly one \(E^\mathfrak{A}\)-class with exactly \(n\) elements. Prove that \(Th(\mathfrak{A})\) has exactly one countable model with infinitely many infinite equivalence classes.

2. a) Let \(T\) be a theory in a language \(L\). Assume that whenever \(\theta_1\) and \(\theta_2\) are universal sentences of \(L\) and \(T \models (\theta_1 \lor \theta_2)\) then either \(T \models \theta_1\) or \(T \models \theta_2\). Prove that for any \(\mathfrak{A}, \mathfrak{B} \models T\) there is some \(\mathfrak{C} \models T\) such that both \(\mathfrak{A}\) and \(\mathfrak{B}\) can be embedded in \(\mathfrak{C}\). [Recall that \(\theta\) is universal iff it has the form \(\forall x_1 \ldots \forall x_n \varphi\) where \(\varphi\) is an open formula]

b) Let \(T\) be an \(\omega\)-categorical theory in a countable language \(L\). Prove that every uncountable model of \(T\) is \(\omega\)-saturated.

3. a) Let \(T\) be a complete theory in a countable language \(L\). Let \(\mathfrak{A}\) be a countable \(\omega_1\)-universal model of \(T\). Prove that there is some \(\omega\)-saturated \(\mathfrak{B}\) such that \(\mathfrak{B} < \mathfrak{A}\).

b) Let \(T\) be a complete theory in a countable language \(L\) and let \(\Phi(x)\) be an \(L\)-type. Assume that \(\Phi\) is realized by at most two elements in every model of \(T\). Prove that there is some formula \(\varphi(x)\) of \(L\) such that for every \(\mathfrak{A} \models T\), \(\Phi^\mathfrak{A} = \varphi^\mathfrak{A}\).
4. a) Assume that $R \subseteq \omega \times \omega$ is r.e. but not recursive and that $R_k \cap R_l = \emptyset$ for all $k \neq l$. Prove that $(\omega \setminus \bigcup_{k \in \omega} R_k)$ is infinite.

b) Prove that $\{ \sigma^\infty : \sigma$ is an open sentence and $\mathcal{M} \models \sigma \}$ is recursive.

5. a) Let $A, B \subseteq \omega$ be recursively inseparable r.e. sets. Assume that $A \leq_m C$ for some $C \subseteq \omega$. Prove that $(\omega \setminus C)$ contains an infinite r.e. subset.

b) Let $f, g$ be total recursive functions of one argument. Let $I_f = \{ e \in \omega : \{e\} = f \}$ and $I_g = \{ e \in \omega : \{e\} = g \}$.

Prove that $I_f =_m I_g$.

6. a) Let $R \subseteq \omega \times \omega$ be r.e. Let $A = \{ k \in \omega : R_k$ is cofinite}. Prove that $A$ is arithmetic.

b) Prove that there are infinitely many $e \in \omega$ such that $\{e\}(2e) = 3e$. 


1. Let $L$ be a countable language and let $\{T_n\}_{n\in \omega}$ be $L$-theories such that $T_n \subseteq T_{n+1}$ for all $n \in \omega$. Let $T^* = \bigcup_{n\in \omega} T_n$ and let $\Phi(x)$ be an $L$-type.

Prove or disprove (with a counterexample) each of the following.

a) If each $T_n$ has a model realizing $\Phi$ then $T^*$ has a model realizing $\Phi$.

b) If each $T_n$ has a model omitting $\Phi$ then $T^*$ has a model omitting $\Phi$.

2. a) Let $T$ be a theory in a language $L$ and let $\mathfrak{B}$ be an $L$-structure. Assume that whenever $\theta$ is a universal sentence of $L$ and $T \models \theta$ then $\mathfrak{B} \models \theta$. Prove that $\mathfrak{B}$ can be embedded in some model of $T$. [Recall that $\theta$ is universal iff it has the form $\forall x_1 \ldots \forall x_n \varphi$ where $\varphi$ is an open formula]

b) Let $T$ be a complete theory of $L$. Assume that $T$ has some model which realizes just finitely many complete types in one variable. Prove that every model of $T$ realizes just finitely many complete types in one variable.

3. a) Let $T$ be a complete theory in a countable language $L$ and let $\Phi(x)$ be an $L$-type. Assume that any two countable models of $T$ omitting $\Phi$ are isomorphic. Prove that every countable model of $T$ omitting $\Phi$ is prime. [Warning: you are not given that $T$ has a prime model]

b) Recall that a countable model $\mathfrak{A}$ is $\omega$-homogeneous iff for all $n \in \omega$ and all $a_0, \ldots, a_n, b_0, \ldots, b_n \in A$ there is an automorphism $h$ of $\mathfrak{A}$ such that $h(a_i) = b_i$ for all $0 \leq i \leq n$ whenever $tp_\mathfrak{A}(a_0, \ldots, a_n) = tp_\mathfrak{A}(b_0, \ldots, b_n)$.
Let $T$ be a complete theory of a countable language $L$, and assume that $\mathfrak{A} \models T$ is countable, $\omega$-homogeneous, and $\omega_1$-universal. Prove that $\mathfrak{A}$ is $\omega$-saturated.

4. a) Let $f : \omega \rightarrow \omega$ be a (total) function. Assume that there is some finite $X \subseteq \omega$ such that for all $n \in (\omega \setminus X)$ we have $f(n + 1) = f(n) + 1$. Prove or disprove (with a counterexample): $f$ is recursive.

b) Let $T$ be a recursively axiomatizable theory containing the axioms for $Q$ such that $\mathfrak{A} \models T$. Prove that there is some formula $\varphi(x)$ (of the language for $Q$) such that $T \vdash \varphi(n)$ for all $n \in \omega$ but $T \not\vdash \forall x \varphi(x)$.

5. a) Let $A \subseteq \omega$ be infinite and r.e. Prove that there are infinite recursive sets $B_0, B_1 \subseteq A$ such that $(B_0 \cap B_1) = \emptyset$.

b) Define sets $A, B \subseteq \omega$ such that $A$ is r.e. in $B$ but $(\omega \setminus A)$ is not r.e. in $(\omega \setminus B)$. [You must prove the sets you define have these properties]

6. a) Let $I = \{ e : |W_e| = 1 \}$. Prove that $A \leq_m I$ for every r.e. $A \subseteq \omega$.

b) Prove that there is some $n \in \omega$ such that $W_n$ is the set whose only element is $n$. 
1. a) Prove or disprove: \(\{1\}\) is definable (by an \(L\)-formula) in the structure \((\mathbb{Q}, <, +)\) for the language \(L\) with \(L_{\text{nl}} = \{<, +\}\).

b) Assume that \(\{T_n : n \in \omega\}\) is a sequence of consistent theories in a language \(L\) such that \(T_n \subseteq T_{n+1}\) for all \(n \in \omega\) and \(T_n \not\models T_{n+1}\) for all \(n \in \omega\). Prove that \(T^* = \bigcup_{n \in \omega} T_n\) is a consistent theory and that \(T^*\) is not finitely axiomatizable.

2. a) Let \(L\) be the language whose only non-logical symbol is the binary relation symbol \(E\). An \(L\)-structure \(\mathfrak{A}\) is called a graph provided \(\mathfrak{A} \models \forall x \forall y (Exy \rightarrow Eyx)\) and \(\mathfrak{A} \models \forall x \neg Exx\).

A graph \(\mathfrak{A}\) is connected iff for all \(a \neq a^*\) in \(A\) either \(E^g(a, a^*)\) holds or there are \(a_1, \ldots, a_n \in A\) for some positive integer \(n\) such that \(E^g(a, a_1), E^g(a_i, a_{i+1})\) for all \(1 \leq i < n\), and \(E^g(a_n, a^*)\) all hold. Prove or disprove each of the following:

a) Every elementary substructure of a connected graph \(\mathfrak{A}\) is connected.

b) Every elementary extension of a connected graph \(\mathfrak{A}\) is connected.

3. a) Let \(T\) be a complete theory in a countable language \(L\) which has a prime model \(\mathfrak{A}\). Assume further that \(\mathfrak{A}\) realizes every \(L\)-type (in finitely many variables) consistent with \(T\). Prove that \(T\) is \(\omega\)-categorical.
b) Let $T$ be a complete theory in a countable language $L$ and let $\Phi(x)$ be an $L$-type consistent with $T$ which is omitted in some model of $T$. Prove that $\Phi$ is realized by infinitely many elements in some model of $T$.

4. a) Let $L$ be the language with $L^{\mu l} = \{+,-,>,\leq,0,s\}$ and let $\mathcal{N} = (\omega,+,\cdot,\leq,0,s)$. Let $T$ be a recursively axiomatizable $L$-theory such that $\mathcal{N} \models T$, let $\varphi(x)$ be a $\Sigma$-formula of $L$, and let $D = \varphi^{\mathcal{N}}$. Assume that $D$ is not recursive. Prove that there is some $\mathcal{A} \models T$ and some $n \in (\omega \setminus D)$ such that $\mathcal{A} \models \varphi(n)$.

b) Let $A, B \subseteq \omega$ be disjoint r.e., non-recursive sets. Prove that $(A \cup B)$ is not recursive.

5. a) Let $R \subseteq (\omega \times \omega)$ be r.e., and assume that $R_k$ is infinite for all $k \in \omega$. Prove that there is some recursive $C \subseteq \omega$ such that $(C \cap R_k) \neq \emptyset$ for all $k \in \omega$ and such that $(\omega \setminus C)$ is infinite.

b) Prove that there is some $f : \omega \rightarrow \omega$ such that for every recursive $g : \omega \rightarrow \omega$ there is some $n \in \omega$ such that $g(k) < f(k)$ for all $k \geq n$.

6. a) Let $A = \{e : \{e\}(k) = 0 \text{ for all } k \in \omega\}$ and let $B = \{e : \{e\}(k) = 1 \text{ for all } k \in \omega\}$. Prove that $A \equiv_m B$.

b) Let $\mathcal{N}$ be the standard model for arithmetic on the natural numbers, and let $T = \{\sigma^{-1} : \mathcal{N} \models \sigma\}$. Prove that $A \leq_m T$ for every arithmetic set $A$. 
1. a) Let $L$ be a language containing (at least) the binary relation symbol $E$. Let $\mathfrak{A}$ be an $L$-structure such that $E^\mathfrak{A}$ is an equivalence relation on $A$. Prove that every $E^\mathfrak{A}$-equivalence class is finite iff every proper elementary extension $\mathfrak{B}$ of $\mathfrak{A}$ contains an element which is not $E^\mathfrak{B}$-equivalent to any element of $\mathfrak{A}$.

b) Let $T$ be a theory in a language $L$ and let $\Phi(x)$ and $\Psi(y)$ be $L$-types. Assume that no model of $T$ realizes both $\Phi(x)$ and $\Psi(y)$. Prove that there is some $\theta \in Sn_L$ such that whenever $\mathfrak{A} \models T$ and $\mathfrak{A}$ realizes $\Phi(x)$ then $\mathfrak{A} \models \theta$, and whenever $\mathfrak{A} \models T$ and $\mathfrak{A}$ realizes $\Psi(y)$ then $\mathfrak{A} \models \neg \theta$.

2. a) Let $\mathfrak{A}$ be an $L$-structure. Assume that $Th(\mathfrak{A})$ is axiomatized by some $\Sigma \subseteq Sn_{L(\mathfrak{A})}$ such that every sentence in $\Sigma$ is either universal or the negation of a universal sentence. Prove that $Th(\mathfrak{A})$ is axiomatized by some $\Sigma^* \subseteq Sn_{L(\mathfrak{A})}$ consisting solely of universal sentences. [Recall that $\theta$ is universal iff it has the form $\forall x_0 \ldots \forall x_k \varphi$ where $\varphi$ is an open formula.]

b) Let $T$ be a complete theory in a countable language $L$. Assume that there is some complete non-principal 1-type consistent with $T$. Prove that every model of $T$ realizes infinitely many complete 1-types.

3. Let $\mathfrak{A}$ be an $L$-structure and let $\Phi(x)$ be a complete $L$-type. Assume that $\Phi(x)$ is realized by exactly three elements in $\mathfrak{A}$. 

a) Assuming, in addition, that \( \Phi(x) \) is principal, prove that \( \Phi(x) \) is realized by exactly three elements in every \( L \)-structure \( \mathfrak{B} \) elementarily equivalent to \( \mathfrak{A} \).

b) Assuming, in addition, that \( \mathfrak{A} \) is \( \omega \)-saturated (but not that \( \Phi \) is principal), prove that \( \Phi(x) \) is realized by exactly three elements in every \( L \)-structure \( \mathfrak{B} \) elementarily equivalent to \( \mathfrak{A} \).

c) Give an example of \( L \), \( L \)-structures \( \mathfrak{A} \) and \( \mathfrak{B} \), and a complete \( L \)-type \( \Phi(x) \) such that \( \Phi(x) \) is realized by exactly three elements in \( \mathfrak{A} \) and \( \mathfrak{A} \equiv \mathfrak{B} \), but \( \Phi(x) \) is not realized by exactly three elements in \( \mathfrak{B} \).

4. a) Let \( S \subseteq (\omega \times \omega) \) be r.e., and assume that \( \bigcup_{k \in \omega} S_k \) is recursive. Prove that there is some recursive \( R \subseteq (\omega \times \omega) \) such that \( R_k \subseteq S_k \) for all \( k \in \omega \) and \( \bigcup_{k \in \omega} R_k = \bigcup_{k \in \omega} S_k \).

b) Let \( T \) be a consistent theory in a language with just finitely many non-logical symbols, including at least the unary function symbol \( s \) and the constant \( 0 \). Assume that every recursive relation is representable in \( T \). Prove that \( T \) is undecidable.

5. a) Let \( A_0 = \{ e \in \omega : \forall k(\{ e \}(k) = 0) \} \) and \( A_1 = \{ e \in \omega : \forall k(\{ e \}(k) = 1) \} \). Prove or disprove: there is some recursive \( B \subseteq \omega \) such that \( A_0 \subseteq B \) and \( (A_1 \cap B) = \emptyset \).

b) Let \( A, B \subseteq \omega \). Explicitly define some \( C \subseteq \omega \) such that the Turing degree of \( C \) is the least upper bound of the Turing degree of \( A \) and the Turing degree of \( B \). You must prove that \( C \) has these properties.

6. a) Recall that \( \text{INF} = \{ e \in \omega : W_e \text{ is infinite} \} \). Prove that \( \text{INF} \leq_m \{ e \in \omega : \forall k(\{ e \}(k) = 0) \} \).

b) Define \( E \subseteq (\omega \times \omega) \) by \( E = \{ (e_1, e_2) : \{ e_1 \} = \{ e_2 \} \} \). Place \( E \) in the arithmetic hierarchy, that is determine (with proof) some \( n \in \omega \) such that either \( E \in \Sigma_n \) or \( E \in \Pi_n \).
1. a) Let $L$ be a language containing (at least) the unary function symbol $s$. An $L$-structure $\mathfrak{A}$ is periodic iff for every $a \in A$ there is some positive integer $n$ such that $(s^A)^n(a) = a$. Prove that there is no $L$-theory $T$ such that for all $L$-structures $\mathfrak{A}$, $\mathfrak{A} \models T$ iff $\mathfrak{A}$ is periodic.

b) Let $T$ be a complete $\omega$-categorical theory in a countable language $L$. Let $\varphi(x, y) \in Fm_L$ and let $\mathfrak{A}$ be any model of $T$. Prove that there is some $n \in \omega$ such that for every $a \in A$ either $|\varphi^\mathfrak{A}(x, a)| < n$ or $\varphi^\mathfrak{A}(x, a)$ is infinite.

2. a) Let $L$ be the language with $L^{\text{ul}} = \{+,-,\cdot,\leq,0,s\}$, let $\mathfrak{N} = (\omega, +, -, \cdot, \leq,0,s)$, and let $\mathfrak{A}$ be any proper elementary extension of $\mathfrak{N}$. Let $\varphi(x) \in Fm_L$. Prove that $\varphi^\mathfrak{A}$ is infinite if and only if there is some $a \in A$ such that $a \in (\varphi^\mathfrak{A} \setminus \omega)$.

b) Let $T$ be a complete theory in a countable language $L$. Let $\Phi(x)$ and $\Psi(x)$ be types consistent with $T$. Assume that every model of $T$ realizes either $\Phi$ or $\Psi$ (or both). Prove that either every model of $T$ realizes $\Phi$ or every model of $T$ realizes $\Psi$.

3. Let $T$ be a complete theory in a countable language $L$ with infinite models.

a) Prove that every countable model of $T$ has a proper countable elementary extension.
b) Assume that $\mathfrak{A} \models T$ is countable and $\omega_1$-universal. Prove that $\mathfrak{A}$ is isomorphic to some proper elementary extension of itself.

c) Assume that $\mathfrak{A} \models T$ is countable and isomorphic to every countable elementary extension of itself. Prove that $\mathfrak{A}$ is $\omega$-saturated.

4. Let $L$ be the language with $L^{al} = \{+, \cdot, <, 0, s\}$ and let $\mathfrak{M} = (\omega, +, \cdot, <, 0, s)$.

   a) Define the function $\pi : \omega \to \omega$ by $\pi(n) =$ the number of primes $\leq n$. Prove or disprove: there is some $\varphi(x, y) \in \text{Fm}_L$ which defines the graph of $\pi$ (that is, the relation $\pi(n) = l$) in $\mathfrak{M}$.

   b) Prove that there is some $\theta(y) \in \text{Fm}_L$ such that for every $\Sigma$-formula $\varphi(x)$ and for every $n \in \omega$ we have $\mathfrak{M} \models \theta([\varphi(\bar{n})])$ iff $\mathfrak{M} \models \varphi(\bar{n})$.

5. a) Assume that $R \subseteq \omega \times \omega$ is r.e., $R_k$ is infinite for all $k \in \omega$, and $(R_k \cap R_l) = \emptyset$ whenever $k \neq l$. Prove that there is some recursive $C \subseteq \omega$ such that $|C \cap R_k| = 1$ for all $k \in \omega$.

   b) Give an example of a theory $T$ in a language $L$ with just finitely many non-logical symbols which is undecidable but not essentially undecidable (you must establish these properties of $T$).

6. a) Prove or disprove: there is some arithmetic relation $R \subseteq \omega \times \omega$ such that for every arithmetic $X \subseteq \omega$ there is some $k \in \omega$ such that $X = R_k$.

   Let $A = \{e \in \omega : 0 \in W_e\}$, $B = \{e \in \omega : 1 \in W_e\}$, and let $C = \{e \in \omega : 0 \not\in W_e\}$. Prove that

   b) $A \leq_m B$, but

   c) $A \not\leq_m C$. 

1. a) Let $T$ be a theory in a language $L$ and let $\varphi(x), \psi_k(x) \in Fm_L$ for all $k \in \omega$. Assume that $T \models \forall x (\psi_k \rightarrow \psi_{k+1})$ for all $k \in \omega$. Assume further that for every $A \models T$ and every $a \in A$ we have

$$A \models \varphi(a) \iff \text{there is some } k \in \omega \text{ such that } A \models \psi_k(a).$$

Prove that there is some $k \in \omega$ such that $T \models \forall x (\varphi \leftrightarrow \psi_k)$.

b) Prove that there is some $A \equiv (\omega, <)$ such that $(\mathbb{R}, <)$ can be isomorphically embedded into $A$.

2. a) Let $L$ be the language whose only non-logical symbol is a binary relation symbol $<$ and let $\mathfrak{B}$ be the $L$-structure $(\mathbb{Q}, <)$. Let $X \subseteq \mathbb{Q}$ be finite. Prove that the set $\mathbb{Z}$ is not definable in the $L(X)$-structure $\mathfrak{B}_X$.

b) Let $L$ be the language whose only non-logical symbol is a binary relation symbol $E$. Let $\mathfrak{A}$ be the $L$-structure such that

$E^\mathfrak{A}$ is an equivalence relation on $A$,

there is exactly one $n$-element equivalence class for every positive integer $n$, and

there are no infinite equivalence classes.

Is there is some proper substructure $\mathfrak{B}$ of $\mathfrak{A}$ such that $\mathfrak{A} \equiv \mathfrak{B}$? Prove or disprove.

3. a) Let $T$ be a complete theory in a countable language $L$. Assume that $T$ has no countable $\omega$-saturated model. Prove that every type consistent with $T$ is realized on at least two non-isomorphic countable models of $T$. 

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b) Let $T$ be a complete theory in a countable language $L$. Let $\Phi(x)$ be a complete non-principal type consistent with $T$. Let $\mathcal{A}$ be an $\omega$-saturated model of $T$. Prove that $\Phi$ is realized by infinitely many elements of $A$.

4. a) Let $T$ be a consistent recursively axiomatizable theory in the language $L$ for arithmetic, let $\varphi(x) \in Fm_L$, and let $A \subseteq \omega$. Assume that $A$ is weakly representable in $T$ by $\varphi$ and $A$ is not recursive. Prove that there is some $k \in \omega$ such that $k \notin A$, $T \not\vdash \varphi(k)$, and $T \not\vdash \varphi(k)$.

b) Let $L$ be a language with just finitely many non-logical symbols which contains at least the unary function symbol $s$ and the constant symbol $\bar{0}$. Let $T$ be a consistent theory of $L$ such that all recursive functions and relations are representable in $T$. Prove that $T$ is undecidable.

5. a) Let $A \subseteq \omega$ be an infinite r.e. set. Prove that there are infinite recursive sets $B_0$ and $B_1$ contained in $A$ such that $(B_0 \cap B_1) = \emptyset$.

b) Let $A, B \subseteq \omega$. Prove that $B$ is r.e. in $A$ iff $B \leq_m A'$.

6. a) Let $A = \{e \in \omega : \{e\}(e) = e\}$. Prove that $A$ is not recursive.

b) Let $A = \{e \in \omega : |W_e| \leq 1\}$ and let $B = \{e \in \omega : |W_e| \geq 2\}$. Prove that $A \equiv_T B$ but $A \not\equiv_m B$. 


1. a) Let $T$ be a theory in a language $L$ containing at least the binary relation symbol $E$. Assume that for every $\mathcal{A} \models T$, $E^\mathcal{A}$ is an equivalence relation on $A$. Assume further that whenever $\mathcal{A} \models T$, $\mathcal{A} \prec \mathcal{B}$, and $a \in A$ then \{\(b \in B : E^\mathcal{B}(a, b)\) holds\} \(\subseteq A\). Prove that there is some $n \in \omega$ such that for every $\mathcal{A} \models T$ every $E^\mathcal{A}$-class has $< n$ elements.

b) Let $T$ be a theory of $L$ and let $\Phi(x)$ and $\Psi(x)$ be $L$-types. We say that a formula $\theta(x)$ of $L$ separates $\Phi$ and $\Psi$ if in every model of $T$ every element realizing $\Phi$ satisfies $\theta$ and every element realizing $\Psi$ satisfies $\neg \theta$. Assume that no formula of $L$ separates $\Phi$ and $\Psi$. Prove that $T$ has a model realizing ($\Phi \cup \Psi$).

2. a) Prove that there is no formula $\varphi(x)$ which defines \{1\} in the structure $(\mathbb{Q}, <, +)$.

b) Prove or disprove: $\text{Th}(\langle \mathbb{Q}, +, \cdot, <, 0, 1 \rangle)$ has a countable $\omega$-saturated model.

3. a) Let $T$ be a complete theory in a countable language. Assume that there is some complete, non-principal type in one variable consistent with $T$. Prove that there are infinitely many complete types in one variable consistent with $T$.

b) Let $L$ be the language whose only non-logical symbol is the binary relation symbol $\prec$. An $L$-structure $\mathcal{A}$ is a linear order provided $\prec^\mathcal{A}$ is a linear order of $A$. Prove that there is some infinite linear order $\mathcal{A}$ such that every $L$-sentence true on $\mathcal{A}$ is also true on some finite linear order.
4. a) Let $A \subseteq \omega$ be an infinite r.e. set. Prove that there is some infinite recursive set $B \subseteq A$.

b) Let $L$ be the language for arithmetic on the natural numbers, that is, $L^\text{nl} = \{+, \cdot, <, 0, s\}$. Let $A = \{[\sigma] : \models \sigma\}$. Prove that $A$ is an $m$-complete r.e. set.

5. a) Let $A = \{e \in \omega : W_e = \emptyset\}$ and let $B = \{e \in \omega : W_e = \omega\}$. Prove that $A$ and $B$ are recursively inseparable, that is there is no recursive $C \subseteq \omega$ such that $A \subseteq C$ and $(B \cap C) = \emptyset$.

b) Prove that there is some $B \subseteq \omega$ such that $A \leq_m B$ for every arithmetic set $A \subseteq \omega$.

6. a) Define a partial recursive function $g$ of one argument which cannot be extended to a total recursive function, i.e., there is no total recursive $f : \omega \rightarrow \omega$ such that $f(n) = g(n)$ whenever $g(n) \downarrow$.

b) Prove that there are infinitely many $e \in \omega$ such that $\{e\}(e+1) = 2e$. 

1. a) Let $L$ be a countable language containing at least the binary relation symbol $E$. Let $T$ be a theory of $L$ such that in every model $\mathfrak{A}$ of $T$, $E^\mathfrak{A}$ is an equivalence relation on $A$. Let $\varphi(x) \in Fm_L$. Assume that no model $\mathfrak{A}$ of $T$ contains an element satisfying $\varphi$ whose $E^\mathfrak{A}$-class is infinite. Prove that there is some $n \in \omega$ such that no model $\mathfrak{A}$ of $T$ contains an element satisfying $\varphi$ whose $E^\mathfrak{A}$-class has $> n$ elements.

b) Let $\mathfrak{A} = (\omega, +, \cdot)$ and let $\mathfrak{B}$ be a proper elementary extension of $\mathfrak{A}$. Prove that there are infinitely many primes in $(B \setminus \omega)$. [An element $b$ of $B$ is prime if it cannot be expressed in $\mathfrak{B}$ as the product of two elements of $B$ each of which is different than $b$]

2. a) Let $L^{nl} = \{ E \}$ where $E$ is a binary relation symbol. Let $\mathfrak{A}$ be the $L$-structure such that $E^\mathfrak{A}$ is an equivalence relation on $A$ with exactly one $n$-element equivalence class for every positive integer $n$ and with no infinite equivalence classes. Let $\mathfrak{B}$ be a countable elementary extension of $\mathfrak{A}$. Prove that $tp_{\mathfrak{B}}(b_1) = tp_{\mathfrak{B}}(b_2)$ for all $b_1, b_2 \in (B \setminus A)$.

b) Let $L = (L_1 \cap L_2)$ and assume that $(L_i \setminus L)$ contains just constant symbols, for $i = 1, 2$. Let $T$ be a complete theory of $L$ and let $T_i$ be a theory of $L_i$ for $i = 1, 2$. Assume that some model of $T$ can be expanded to a model of $T_1$, and also that some model of $T$ can be expanded to a model of $T_2$. Prove that there is some model $\mathfrak{A}$ of $T$ such that $\mathfrak{A}$ can be expanded to a model $\mathfrak{A}_1$ of $T_1$ and $\mathfrak{A}$ can also be expanded to a model $\mathfrak{A}_2$ of $T_2$. 

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3. a) Let $L$ be a countable language containing at least the binary relation symbol $E$. Let $T$ be a theory of $L$ such that $T \models \forall x \forall y (Exy \rightarrow Eyx)$.

If $\mathfrak{A} \models T$ and $a, a^* \in A$ with $a \neq a^*$ we say that $a, a^*$ are connected if either $E^\mathfrak{A}(a, a^*)$ holds or there are $a_1, \ldots, a_n \in A$ for some positive integer $n$ such that

$$E^\mathfrak{A}(a, a_1), E^\mathfrak{A}(a_i, a_{i+1}) \text{ for all } 1 \leq i < n, \text{ and } E^\mathfrak{A}(a_n, a^*)$$

all hold. Assume that in every model of $T$ there is a pair of distinct elements that is not connected. Prove that there is some $\psi(x, y) \in FM_L$ consistent with $T$ such that for every $\mathfrak{A} \models T$ and every $a, a^* \in A$, if $\mathfrak{A}_a \models \psi(\bar{a}, \bar{a}^*)$ then $a \neq a^*$ and $a, a^*$ are not connected.

b) Let $T$ be a complete theory in a countable language $L$. Let $\mathfrak{A}$ be a prime model of $T$ and let $\Phi(x)$ be a complete type of $L$. Assume that $\Phi$ is realized by exactly two elements in $\mathfrak{A}$. Prove that $\Phi$ is realized by exactly two elements in every model of $T$.

4. a) Let $R \subseteq \omega \times \omega$ be r.e. and assume that $\bigcup_{k \in \omega} R_k$ is recursive. Prove that there is some recursive $S \subseteq \omega \times \omega$ such that $S_k \subseteq R_k$ for all $k \in \omega$ and $\bigcup_{k \in \omega} S_k = \bigcup_{k \in \omega} R_k$.

b) A total function $f : \omega \rightarrow \omega$ is monotone iff for all $m, n \in \omega$, if $m \leq n$ then $f(m) \leq f(n)$. Let $f$ be a recursive monotone function. Prove that the range of $f$ is recursive. [Warning: $f$ need not be strictly increasing]

5. a) Give an example of a theory $T$ which is undecidable but not essentially undecidable. [You must prove both assertions about $T$]

b) Prove that there are r.e. sets $A, B \subseteq \omega$ such that $(A \cap B) = \emptyset$ but there is no recursive $C \subseteq \omega$ such that $A \subseteq C$ and $(B \cap C) = \emptyset$.

6. a) Prove that $\{e : 2 \in W_e\} \equiv_m \{e : 3 \in W_e\}$.

b) Let $I = \{e \in \omega : W_e = \{3\}\}$. Determine some $n \in \omega$ such that either $I \in \Sigma_n$ or $I \in \Pi_n$. [You need not prove your choice of $n$ is minimal]
1. a) Let $T$ be a theory of a language $L$. Assume that there is some $\theta \in S_{nL}$, such that for every model $\mathfrak{A}$ of $T$, $\mathfrak{A}$ is infinite iff $\mathfrak{A} \models \theta$. Prove that there is some $n \in \omega$ such that every finite model of $T$ has at most $n$ elements.

b) Prove that $(\mathbb{Q}, +, \cdot, 0, 1)$ is a prime model of its complete theory.

2. a) Let $\mathfrak{M} = (\omega, +, \cdot, <, 0, s)$ be the standard model for arithmetic on $\omega$ and let $\mathfrak{B}$ be some fixed proper elementary extension of $\mathfrak{M}$. Let $\varphi(x) \in Fm_L$ and assume that $\varphi^{\mathfrak{M}} = \varphi^{\mathfrak{B}}$. Prove that $\varphi^{\mathfrak{M}}$ is finite.

b) Let $L^{nd} = \{E\}$ where $E$ is a binary relation symbol. An $L$-structure $\mathfrak{A}$ is a graph provided $\mathfrak{A} \models \forall x \forall y (Exy \rightarrow Eyx)$. A graph $\mathfrak{A}$ is connected iff for all $a, a^* \in A$ with $a \neq a^*$ there are $a_1, \ldots, a_n \in A$ for some $n \in \omega$ such that $E^\mathfrak{A}(a, a_1), E^\mathfrak{A}(a_i, a_{i+1})$ for all $1 \leq i < n$, and $E^\mathfrak{A}(a_n, a^*)$ all hold. Let $T$ be an $L$-theory such that every connected graph is a model of $T$. Prove that there is some graph which is a model of $T$ but is not connected.

3. a) Let $T$ be a complete theory in a countable language $L$. Assume that for every $\varphi(x) \in Fm_L$ consistent with $T$ there is some $\psi(x) \in Fm_L$ such that both $(\varphi \land \psi)$ and $(\varphi \land \neg \psi)$ are consistent with $T$. Prove that $T$ does not have a prime model.
b) Let $T$ be a complete theory in a countable language $L$. Let $\mathfrak{A} \models T$ be countable and assume that $\mathfrak{A}$ is isomorphic to each of its countable elementary extensions. Prove that $T$ has a countable $\omega$-saturated model and that $\mathfrak{A}$ itself is $\omega$-saturated.

4. a) Let $L$ be a language with just finitely many non-logical symbols, including at least the unary function symbol $s$ and the constant $0$. Let $T$ be a theory of $L$ such that every recursive relation is representable in $T$. Prove that $T$ is undecidable.

b) Let $A = \{[\sigma] \in \mathcal{S} : \sigma$ is a $\Sigma$-sentence and $\mathfrak{N} \models \sigma\}$, where $\mathfrak{N}$ is the usual model for arithmetic on $\omega$. Prove that $A$ is not $\Pi_1$.

5. a) Let $R \subseteq \omega \times \omega$ be r.e. Assume that $R_k \neq \emptyset$ for all $k \in \omega$, $\bigcup_{k \in \omega} R_k = \omega$, and for all $k, l \in \omega$ either $R_k = R_l$ or $R_k \cap R_l = \emptyset$. Assume further that there is some recursive $C \subseteq \omega$ such that for all $k \in \omega$, $|R_k \cap C| = 1$. Prove that $R$ is recursive.

b) Let $A = \{e \in \omega : W_e$ is either finite or cofinite}. Find an $n$ so that $A \in \Delta_n$. [You need not prove your $n$ is the least possible]

6. a) Let $A, B \subseteq \omega$ be recursively inseparable r.e. sets (so $A \cap B = \emptyset$ and there is no recursive set $A^*$ with $A \subseteq A^*$ and $A^* \cap B = \emptyset$.) Assume that $A \leq_m C$ where $C \subseteq \omega$. Prove that there is some infinite r.e. set $D \subseteq \omega$ such that $C \cap D = \emptyset$.

b) Let $I = \{e \in \omega : |W_e| = 1\}$. Prove that every r.e. set is many-one reducible to $I$. 
1. a) Prove or disprove: \((\mathbb{Z}, +)\) has a proper elementary substructure.

b) Assume that \(\mathfrak{A}\) and \(\mathfrak{B}\) are \(L\)-structures and \(\mathfrak{A} \equiv \mathfrak{B}\). Prove that there is some \(\mathfrak{C}\) such that both \(\mathfrak{A}\) and \(\mathfrak{B}\) can be elementarily embedded in \(\mathfrak{C}\).

2. a) Let \(L\) be a countable language containing (at least) the binary relation symbol \(E\). Let \(T\) be a complete \(\omega\)-categorical \(L\)-theory, let \(\mathfrak{A}\) be a countable model of \(T\), and assume that \(E^\mathfrak{A}\) is an equivalence relation on \(A\). Prove that there is some \(n \in \omega\) such that for every \(a \in A\) the \(E^\mathfrak{A}\)-class of \(a\) is either infinite or has fewer than \(n\) elements.

b) Let \(T\) be a complete theory in a countable language \(L\) and let \(\varphi(x)\) be a complete \(L\)-type. Assume that \(T\) has some model which contains exactly one element realizing \(\varphi\) and also some model which contains exactly two elements realizing \(\varphi\). Prove that \(T\) has a model omitting \(\varphi\).

3. a) Let \(L^{\omega} = \{c_n : n \in \omega\}\). Let \(\mathfrak{A}\) be an \(L\)-structure such that \(c_n^{\mathfrak{A}} \neq c_m^{\mathfrak{A}}\) for all \(n \neq m\) and such that there is exactly one element \(a^* \in A\) such that \(a^* \neq c_n^{\mathfrak{A}}\) for all \(n \in \omega\). Prove that there is no formula \(\varphi(x)\) of \(L\) such that \(\varphi^{\mathfrak{A}} = \{a^*\}\).

b) Let \(\mathfrak{A}\) be a countable \(\omega\)-saturated structure for a countable language \(L\). Let \(a_0 \in A\) be such that \(h(a_0) = a_0\) for every automorphism \(h\) of \(\mathfrak{A}\). Prove that there is some formula \(\varphi(x)\) of \(L\) such that \(\varphi^{\mathfrak{A}} = \{a_0\}\).
4. a) Let $T$ be a recursively axiomatizable theory true on $\mathfrak{N}$, the standard model for arithmetic on the natural numbers. Let $X \subseteq \omega$ be r.e. but not recursive, and assume that $X = \varphi^\mathfrak{N}$ for some $\Sigma$-formula $\varphi(x)$. Prove that there is some $\mathfrak{B} \models T$ such that $\mathfrak{B} \models \varphi(\bar{n})$ for some $n \in (\omega \setminus X)$.

b) Let $R \subseteq (\omega \times \omega)$ be r.e. Assume the $R_n$'s are infinite and pairwise disjoint. Prove that there is some recursive $C \subseteq \omega$ such that $|R_n \cap C| = 1$ for all $n \in \omega$.

5. a) Let $L^{nl} = \emptyset$. Give an example of a theory $T$ of $L$ which is undecidable but all its complete extensions (in $L$) are decidable.

b) Let $T$ be a recursively axiomatizable theory in a language $L$ with just finitely many non-logical symbols. Assume that $T$ has just finitely many complete extensions (in $L$). Prove that $T$ is decidable.

6. a) Recall that
$$F I N = \{e : W_e \text{ is finite}\} \text{ and } I N F = \{e : W_e \text{ is infinite}\}.$$ Prove that $F I N \leq_T I N F$ but $F I N \not\leq_m I N F$.

b) Recall that $R E C = \{e : W_e \text{ is recursive}\}$. Prove that $R E C$ is arithmetic, that is, that $R E C$ is in $\Sigma_n$ or $\Pi_n$ for some $n \in \omega$. Although you should try to make $n$ as small as possible, you do not need to prove your choice of $n$ is minimal.
1. a) Let a theory $T$ and sentences $\sigma_n$ of a language $L$ be given. Assume that $T \models (\sigma_n \rightarrow \sigma_{n+1})$ for all $n \in \omega$. Assume further that for every $\mathfrak{A} \models T$ there is some $n \in \omega$ such that $\mathfrak{A} \models \sigma_n$. Prove that there is some $n_0 \in \omega$ such that $T \models (\sigma_{n_0+1} \rightarrow \sigma_{n_0})$. [In fact, $T \models (\sigma_m \rightarrow \sigma_{n_0})$ will hold for all $m > n_0$.]

b) Let $L_0$ be the language containing just the binary relation symbol $<$, let $L$ be a language containing $L_0$, and let $T$ be a theory of $L$. Assume that $(\omega, <)$ embeds into the $L_0$-reduct of some model of $T$. Prove that $(\mathbb{Q}, <)$ can be embedded into the $L_0$-reduct of some model of $T$.

2. a) Let $\mathfrak{A} = (\omega, +, \cdot, <, 0, s)$. In $\mathfrak{A}$ the set of primes is definable by the following formula $\varphi(x)$:

$$
(s0 < x) \land \forall y \exists z (x = y \cdot z \rightarrow (x = y) \lor (x = z))
$$

Let $\mathfrak{B}$ be any proper elementary extension of $\mathfrak{A}$. Prove that $\mathfrak{B}$ contains a new prime, that is, some element $b$ satisfying $\varphi(x)$ which is not in $\omega$.

b) Let $L$ be the language whose only non-logical symbol is the binary relation $E$ and let $T$ be the $L$-theory axiomatized by sentences saying that $E$ is an equivalence relation on the universe with infinitely many equivalence classes, each of which is infinite. Prove that $T$ is model complete, that is, for all models $\mathfrak{A}$ and $\mathfrak{B}$ of $T$, if $\mathfrak{A} \subseteq \mathfrak{B}$ then $\mathfrak{A} \prec \mathfrak{B}$. 

1
3. a) Let $T$ be a complete theory in a countable language $L$. Assume that there is some non-principal complete type in one variable consistent with $T$. Prove that every model of $T$ realizes (at least) three different complete types in one variable. [In fact each model of $T$ will realize infinitely many, but you need not prove this.]

b) Let $\mathfrak{a}$ be an $\omega$-saturated $L$-structure and let $\varphi(x, y)$ be an $L$-formula. Assume that for every $a \in A$ the set $\varphi^a(x, a)$ is finite. Prove that there is some $n \in \omega$ such that for every $a \in A$ the set $\varphi^a(x, a)$ contains at most $n$ elements.

4. a) Assume that $R \subseteq \omega \times \omega$ is r.e. and that $R_n$ is infinite for every $n \in \omega$. Let $g : \omega \rightarrow \omega$ be any recursive function. Prove that there is some recursive function $f : \omega \rightarrow \omega$ such that $f(n) \in R_n$ and $g(n) < f(n)$ for all $n \in \omega$.

b) Let $L$ be the language whose only non-logical symbol is the binary relation $E$ and let $T_0$ be the $L$-theory axiomatized by sentences stating that $E$ is an equivalence relation on the universe. Prove that $T$ has a complete undecidable extension.

5. a) Define $f : \omega \rightarrow \omega$ by

$$f(n) = (\mu k)[\{n\} = \{k\}].$$

Prove that $f$ is not recursive.

b) Assume that $B \subseteq \omega$ is such that $A \leq_m B$ for all r.e. sets $A$. Prove that $B$ contains some infinite r.e. subset.

6. a) Let $A_n \subseteq \omega$ be given for all $n \in \omega$. Prove that there is some $B \subseteq \omega$ such that $A_n \leq_T B$ holds for all $n \in \omega$.

b) Let $A = \{e \in \omega : \{e\}(5) = 7\}$. Prove that $A \equiv_m K$. [Recall that $K = \{e : \{e\}(e) \downarrow\}]$
1. a) Let $T$ be a theory of $L$, let $\Phi(x)$ and $\Psi(x)$ be types of $L$. Assume that for every $a \models T$ and all $a \in A$, $a$ realizes $\Phi$ iff $a$ does not realize $\Psi$. Prove that there is some $\varphi(x) \in Fm_L$ such that $\Phi^a = \varphi^a$ for every model $a$ of $T$.

b) Let $L$ be a language containing (at least) the binary relation symbol $E$. Let $a$ be an $\omega$-saturated $L$-structure in which $E^a$ is an equivalence relation on $A$ with exactly one infinite equivalence class. Prove that there is some $n \in \omega$ such that every finite $E^a$-class has at most $n$ elements.

2. a) Prove or disprove: $(\omega, +)$ has a proper elementary substructure.

b) Let $T$ be an $L$-theory. Let $a$ be an $L$-structure which cannot be embedded in any model of $T$. Prove that there is an existential sentence $\theta$ of $L$ (that is, $\theta$ has the form $\exists x_1 \ldots \exists x_n \alpha$ where $\alpha$ is an open formula of $L$) such that $a \models \theta$ but $T \models \neg \theta$.

3. a) Prove that the structure $(\omega, |)$ has uncountably many automorphisms (where $n|k$ iff $k = n \cdot l$ for some $l \in \omega$).

b) Let $T$ be a complete theory in a countable language $L$ and let $\Phi(x)$ be an $L$-type which is omitted on some model of $T$. Assume further that any two countable models of $T$ omitting $\Phi$ are isomorphic. Prove that every countable model of $T$ omitting $\Phi$ is prime.

[Warning: You cannot assume that $T$ has a prime model]
4. a) Assume that $R \subseteq \omega \times \omega$ is r.e. and that $\cup_{k \in \omega} R_k = \omega$. Prove that there is some recursive $S \subseteq R$ such that $\cup_{k \in \omega} S_k = \omega$.

b) Let $L$ be a language with only finitely many non-logical symbols and let $L' = L \cup \{c\}$ where $c$ is a constant symbol not in $L$. Let $T'$ be a finitely axiomatizable undecidable theory of $L'$ and let $T = T' \cap S_{n_L}$. Prove that $T$ is also undecidable.

5. Recall that subsets $A$ and $B$ of $\omega$ are called recursively inseparable if there is no recursive $C \subseteq \omega$ such that $A \subseteq C$ and $B \cap C = \emptyset$.

a) Prove that there are disjoint r.e. subsets $A$ and $B$ of $\omega$ which are recursively inseparable.

b) Assume that $A$ and $B$ are disjoint r.e. subsets of $\omega$ which are recursively inseparable. Prove that $\omega \setminus (A \cup B)$ is infinite.

6. a) Let $A = \{[\sigma]: \sigma \in S_{n_L} \text{ and } Q \vdash \sigma\}$ (where $L$ is the usual language for arithmetic on the natural numbers). Prove that $A$ is an $m$-complete r.e. set.

b) Prove that there is some $A \subseteq \omega$ such that $A \in \Sigma_3$ but $A \notin \Pi_3$. 
1. a) Let $T$ be a theory of a language $L$, and let $\varphi_i(x)$ be formulas of $L$ for all $i \in \omega$. Assume that for all $i \in \omega$

$$T \models \forall x(\varphi_{i+1}(x) \rightarrow \varphi_i(x)) \quad \text{and} \quad T \models \neg \forall x(\varphi_i(x) \rightarrow \varphi_{i+1}(x)).$$

Prove that $T$ has a model $\mathfrak{a}$ with an element $a$ such that $\mathfrak{a} \models \varphi_i(a)$ for all $i \in \omega$.

b) Let $T$ be a complete theory in a countable language $L$, and assume that for each $n > 0$ there are just countably many complete types in $n$ free variables consistent with $T$. Prove that $T$ has a prime model.

2. a) Prove or disprove: $(\mathbb{Z}, <)$ has a proper elementary submodel.

b) Does $Th((\mathbb{Z}, +, 1))$ have a countable $\omega$-saturated model? Prove your answer.

3. a) Let $\mathfrak{a}$ be the unique countable model of a a complete $\omega$-categorical theory $T$ in a countable language $L$, and let $\varphi(x, y) \in Fm_L$. Prove that there is some $n \in \omega$ such that for every $a \in A$, either $|\varphi^a(x, a)| < n$ or $\varphi^a(x, a)$ is infinite.

b) Let $T$ be a complete theory in a countable language $L$ having infinite models. Assume that for every $\varphi(x) \in Fm_L$ and for every $\mathfrak{a} \models T$, $\varphi^a$ is either finite or cofinite (meaning its complement is finite). Prove that there is exactly one non-principal complete type $\Phi(x)$ in the single variable $x$ consistent with $T$. 


4. a) Let $T$ be a consistent recursively axiomatized theory containing the axioms for $Q$. Prove that there is a formula $\varphi(x)$ such that $T \models \varphi(n)$ for all $n \in \omega$ but $T \not\models \forall x \varphi(x)$.

b) Let $R \subseteq \omega \times \omega$ be r.e., and assume that $|\omega \setminus R_k| = 2$ for every $k \in \omega$. Prove that $R$ is recursive.

5. a) Assume that $A \subseteq \omega$ is such that
\[
\{e : W_e = 0\} \subseteq A \quad \text{and} \quad \{e : W_e = \omega\} \cap A = \emptyset.
\]
Prove that $A$ is not recursive.

b) Assume that $A \subseteq \omega$ is such that $K \preceq_m A$. Prove that $A$ contains an infinite r.e. subset.

[Recall that $K = \{e : e \in W_e\}$]

6. a) Let $T$ be a consistent, decidable theory in a language $L$ with just finitely many non-logical symbols. Prove that $T \subseteq T^*$ for some complete, decidable theory $T^*$ of $L$.

[Hint: Let $\{\sigma_n : n \in \omega\}$ be a recursive list of all sentences of $L$ ...]

b) Prove that $TOT \equiv_m INF$.

[Recall that $TOT = \{e : W_e = \omega\}$ and $INF = \{e : W_e \text{ is infinite}\}$]
1. a) Assume that $L \subseteq L'$, let $T'$ be an $L'$-theory and let $\mathfrak{a}$ be an $L$-structure. Assume that there is no $\mathfrak{a}' \models T'$ such that $\mathfrak{a}$ is elementarily equivalent to the $L$-reduct of $\mathfrak{a}'$. Prove that there is some $\sigma \in \mathcal{S}_{nL}$ such that $\mathfrak{a} \models \sigma$ and $T' \models \neg \sigma$.

b) Let $L^u = \{ E \}$ where $E$ is a binary relation symbol. Let $K$ be the class of all $L$-structures $\mathfrak{a}$ for which $E^\mathfrak{a}$ is an equivalence relation on $A$ with at least one finite $E^\mathfrak{a}$-class. Prove that there is no theory $T$ of $L$ such that $K = \text{Mod}(T)$.

[Hint: Assume that $K \subseteq \text{Mod}(T)$ and find $\mathfrak{a} \models T$ such that $\mathfrak{a} \notin K$.]

2. a) Let $L$ contain at least the binary relation symbol $E$, and let $\mathfrak{a}$ be an infinite $\omega$-saturated $L$-structure such that $E^\mathfrak{a}$ is an equivalence relation on $A$. Assume that whenever $\mathfrak{a} \prec \mathfrak{b}$ and $a \in A$ then

\[ \{ b \in B : E^\mathfrak{b}(a, b) \text{ holds} \} \subseteq A. \]

Prove that there is some $n_0 \in \omega$ such that every $E^\mathfrak{a}$-class has at most $n_0$ elements.

b) Let $L$ be a countable language containing at least the unary relation symbols $P_n$ for $n \in \omega$, and let $T$ be a theory of $L$. Assume that $T$ has a model $\mathfrak{a}$ such that for every $\varphi(x) \in Fm_L$ if $\varphi^\mathfrak{a} \neq \emptyset$ then there is some $k \in \omega$ such that $(\varphi^\mathfrak{a} \cap P_k^\mathfrak{a}) \neq \emptyset$. Prove that $T$ has a model $\mathfrak{b}$ such that $B = \bigcup_{k \in \omega} P_k^\mathfrak{b}$.

3. Let $T$ be a complete theory in a countable language $L$. Recall that a complete type $\Phi(x)$ consistent with $T$ is said to be non-principal provided it does not contain a complete formula $\varphi(x)$.
a) Assume that $\Phi(x)$ is a non-principal complete type consistent with $T$. Prove that $T$ has some model which contains infinitely many elements realizing $\Phi(x)$.

b) Assume that there are no non-principal complete types $\Phi(x)$ in the single free variable $x$ consistent with $T$. Prove that there are only finitely many complete types in the single free variable $x$ consistent with $T$.

4. a) Let $A$ and $B$ be r.e. subsets of $\omega$. Assume that $(A \cup B)$ is recursive. Prove that there are recursive sets $A' \subseteq A$ and $B' \subseteq B$ such that $(A \cup B) = (A' \cup B')$.

b) Let $A$ be an infinite r.e. subset of $\omega$. Prove that there is an infinite recursive set $B$ with $B \subseteq A$.

5. a) Give a theory $T$ in a language $L$ with just finitely many non-logical symbols which has an r.e. set of axioms but is such that 

$$\{ n \in \omega : T \text{ has a model } \mathfrak{A} \text{ with } |\mathfrak{A}| = n \}$$

is not recursive. Prove that it has these properties.

b) Assume that $R \subseteq \omega \times \omega$ is r.e. Let $A = \{ k \in \omega : R_k \text{ is infinite} \}$. Prove that $A$ is $\Pi_2$.

6. a) Recall that $K = \{ e \in \omega : e \in W_e \}$ and that $\text{INF} = \{ e \in \omega : W_e \text{ is infinite} \}$. Prove that $K \trianglelefteq_m \text{INF}$.

b) Let $\mathcal{F}$ be a non-empty set of partial recursive functions of one argument and let $I = \{ e \in \omega : \{ e \} \in \mathcal{F} \}$. Prove that $I \not\subseteq_m (\omega \setminus I)$. 

1. a) Let $T$ be a theory of a language $L$ containing (at least) the binary relation symbol $E$ and so that for every $\mathfrak{a} \models T$, $E^\mathfrak{a}$ is an equivalence relation on $A$. Assume further that whenever $\mathfrak{a} \models T$, $\mathfrak{a} \prec \mathfrak{b}$, $a \in A$ and $b \in (B \setminus A)$ then $\mathfrak{b}_B \models \neg E(\bar{a}, \bar{b})$. Prove that there is some $n_0 \in \omega$ such that for every $\mathfrak{a} \models T$ all $E^\mathfrak{a}$-classes have $\leq n_0$ elements.

b) Let the only non-logical symbol of $L$ be the binary relation symbol $E$. Let $\mathfrak{a}$ be the $L$-structure in which $E^\mathfrak{a}$ is an equivalence relation on $A$ with infinitely many 2 element classes and infinitely many 3 element classes and no other classes. Let $\mathfrak{a} \subseteq \mathfrak{b}$ where $\mathfrak{b}$ adds exactly one more 2 element class and nothing else. Prove that $\mathfrak{a} \prec \mathfrak{b}$. [Hint: why are $\mathfrak{a}$ and $\mathfrak{b}$ elementarily equivalent?]

2. a) Is the structure $(\mathbb{R}, +, \cdot, 0, 1)$ $\omega$-saturated? Explain.

b) Assume that the $L$-structure $\mathfrak{a}$ realizes exactly three different complete $L$-types in one free variable. Prive that the same is true of every model of $Th(\mathfrak{a})$.

3. a) Let $T$ be a complete theory in a countable language $L$, and let $\Phi(x)$ be an $L$-type. Assume that in every model of $T$ the type $\Phi$ is realized by at most 2 elements. Prove that there is a formula $\varphi(x)$ of $L$ such that for every $\mathfrak{a} \models T$, $\Phi^\mathfrak{a} = \varphi^\mathfrak{a}$. 
b) Let $T$ be a complete theory in a countable language $L$ which has no prime model. Let $\Phi(x)$ be an $L$-type omitted on some model of $T$. Prove that $T$ has at least two nonisomorphic countable models omitting $\Phi$.

4. a) Assume that $R \subseteq \omega \times \omega$ is r.e. and that $R_k$ is infinite for all $k \in \omega$. Prove that there is a strictly increasing recursive function $f$ on $\omega$ such that $f(k) \in R_k$ for all $k \in \omega$.

b) Prove that there is a function $g : \omega \to \omega$ such that for every recursive function $f$ on $\omega$ there is some $n_0 \in \omega$ so that for all $n \geq n_0$ we have $f(n) < g(n)$.

5. a) Assume that $R \subseteq \omega \times \omega$ is r.e. but not recursive and that $\bigcup_{k \in \omega} R_k$ is recursive. Prove that $R_k \cap R_l \neq \emptyset$ for some $k \neq l$.

b) Let $f_1$ and $f_2$ be partial recursive functions and assume that $f_1 \neq f_2$. Let $B_1 = \{ e : \{ e \} = f_1 \}$ and let $B_2 = \{ e : \{ e \} = f_2 \}$. Prove that there is no recursive set $A$ such that $B_1 \subseteq A$ and $B_2 \cap A = \emptyset$.

6. a) Prove that $\{ e : 0 \in W_e \}$ is an $m$-complete r.e. set.

b) Let $\text{REC} = \{ e : W_e \text{ is recursive } \}$. Use Post's Theorem to prove that $\text{REC}$ is r.e. in $\emptyset'$.
1. a) Let a theory $T$ and sentences $\sigma_n$ for $n \in \omega$ be given. Assume that $T \models (\sigma_n \rightarrow \sigma_{n+1})$ and $T \not\models (\sigma_{n+1} \rightarrow \sigma_n)$ for all $n \in \omega$. Prove that $T$ has a model $\mathfrak{A}$ such that $\mathfrak{A} \models \neg \sigma_n$ for all $n \in \omega$.

b) Let $L$ be a language containing at least the binary relation symbol $E$, and let $\mathfrak{A}$ be an $L$-structure so that $E^\mathfrak{A}$ is an equivalence relation on $A$. Assume that for every elementary extension $\mathfrak{B}$ of $\mathfrak{A}$ and every $b \in B$ there is some $a \in A$ such that $E^\mathfrak{A}(a, b)$ holds. Prove that $E^\mathfrak{A}$ has just finitely many equivalence classes.

2. a) Prove that $(Q, \leq)$ is isomorphically embeddable in some $\mathfrak{B} \equiv (\omega, \leq)$.

b) Prove or disprove: $(\mathbb{Z}, +)$ has a proper elementary submodel.

3. a) Let $L$ be a countable language containing at least the binary relation symbol $E$, and let $T$ be a theory of $L$ such that for every model $\mathfrak{A}$ of $T$, $E^\mathfrak{A}$ is an equivalence relation on $A$. Assume that for every model $\mathfrak{A}$ of $T$ some $E^\mathfrak{A}$ class is infinite. Prove that there is some formula $\varphi(x)$ of $L$ consistent with $T$ so that whenever $\mathfrak{A}$ is a model of $T$, $a \in A$ and $\mathfrak{A} \models \varphi(\bar{a})$ then the $E^\mathfrak{A}$-class of $a$ is infinite.

b) Let $T$ be a complete theory in a countable language $L$, let $\Phi(x)$ and $\Psi(x)$ be $L$-types, and let $\mathfrak{A}$ be an $\omega$-saturated model of $T$. Assume that $\Phi^\mathfrak{A} = (A \setminus \Psi^\mathfrak{A})$. Prove that there is some formula $\varphi(x)$ of $L$ such that for every model $\mathfrak{B}$ of $T$, $\Phi^\mathfrak{B} = \varphi^\mathfrak{B}$.
4. a) Let \( R \subseteq \omega \times \omega \) be r.e. and assume that \( R_k \neq \emptyset \) for all \( k \in \omega \) and that \( R_k \cap R_l = \emptyset \) for all \( k \neq l \). Prove that there is some r.e. \( C \subseteq \omega \) such that \( |R_k \cap C| = 1 \) for all \( k \in \omega \).

b) Let \( X \subseteq \omega \) and a formula \( \varphi(x) \) of the language of arithmetic be given. Assume that \( \varphi \) weakly represents \( X \) in every consistent theory \( T \) containing \( Q \). Prove that \( X \) is recursive.

5. a) Let \( T \) be a recursively axiomatizable theory and assume that \( T \) has just finitely many complete extensions (in the same language). Prove that \( T \) is decidable.

b) Define \( f : \omega \to \omega \) by \( f(e) = \) the least \( d \) such that \( \{d\} = \{e\} \). Prove that \( f \) is not recursive.

6. a) Give an example (with proof) of a set \( X \subseteq \omega \) which is \( \Pi_1 \) but not \( \Sigma_1 \).

b) Prove or disprove: \( \{[\sigma] : \mathbb{N} \models \sigma \} \) is arithmetic.
1. a) Let $T$ and $T'$ be theories of $L$ such that for every $L$-structure $A$, $A \models T$ iff $A \not\models T'$. Prove that $T$ is finitely axiomatizable.

b) Prove that every countable linear order can be isomorphically embedded in $(\mathbb{Q}, \leq)$.

2. a) Prove or disprove: $(\mathbb{R} \setminus \{0\}, \leq)$ is an elementary substructure of $(\mathbb{R}, \leq)$.

b) Let $T$ be a complete $\omega$-categorical theory in a countable language $L$. Prove that there is an integer $k$ such that for every model $A$ of $T$ and every formula $\varphi(x)$ of $L$ with just one free variable, if $\varphi^A$ has more than $k$ elements then $\varphi^A$ is infinite.

3. Let $T$ be a complete theory in a countable language $L$, let $A$ be an $\omega$-saturated model of $T$, and let $\Phi(x)$ be a type in one free variable consistent with $T$. Assume that $\Phi$ is realized in $A$ by exactly two elements of $A$. Prove that $\Phi$ is realized by exactly two elements in every model of $T$.

4. a) Assume that $R \subseteq \omega \times \omega$ is r.e. and $\bigcup_{k \in \omega} R_k = \omega$. Prove that there is some recursive $S \subseteq R$ such that $\bigcup_{k \in \omega} S_k = \omega$ and $S_k \cap S_l = \emptyset$ whenever $k \neq l$. 


b) Let $T$ be a consistent recursively axiomatizable extension of the theory $Q$. Find a formula $\varphi(x)$ such that $T \models \varphi(\bar{n})$ for all $n \in \omega$ but $T \not\models \forall x \varphi(x)$. (Be sure to show that the formula you define has this property.)

5. a) Let $L$ be a language with just finitely many non-logical symbols and let $L' = L \cup \{c\}$ where $c$ is a constant symbol not in $L$. Assume that $T'$ is a finitely axiomatizable essentially undecidable theory of $L'$ and let $T = T' \cap S_{n_L}$. Prove that $T$ is essentially undecidable.

b) Prove that $A = \{e \in \omega : \{e\}(e) = e\}$ is not recursive.

6. An r.e. set $\mathcal{A} \subseteq \omega$ is said to be simple if $(\omega \setminus \mathcal{A})$ is infinite but does not contain an infinite r.e. subset.

a) Prove that the intersection of two simple r.e. sets is simple.

b) Show that $K = \{e : e \in W_e\}$ is not simple.
1. a) Let $L$ be a language containing at least the binary relation symbol $E$ and let $T$ be a theory of $L$ so that in every model $\mathfrak{a}$ of $T$, $E^\mathfrak{a}$ is an equivalence relation on $A$. Assume that in every model $\mathfrak{a}$ of $T$, every $E^\mathfrak{a}$-class is finite. Prove that there is some $n \in \omega$ so that in every model $\mathfrak{a}$ of $T$, every $E^\mathfrak{a}$-class contains at most $n$ elements.

b) Let $\Sigma_1$ and $\Sigma_2$ be sets of sentences of $L$ such that there is no sentence $\theta$ of $L$ so that $\Sigma_1 \models \theta$ and $\Sigma_2 \models \neg \theta$. Prove that $(\Sigma_1 \cup \Sigma_2)$ has a model.

2. a) Let $\mathfrak{a}$ be an $L$-structure and let $\varphi(x)$ be a formula of $L$. Prove that $\varphi^\mathfrak{a}$ is finite iff there is no $\mathfrak{b}$ so that $\mathfrak{a} \prec \mathfrak{b}$ and $\varphi^\mathfrak{a} \neq \varphi^\mathfrak{b}$.

b) Let $\{\varphi_i(x) : i \in \omega\}$ be an infinite set of $L$-formulas and let $\mathfrak{a}$ be an $\omega$-saturated $L$-structure. Assume that for every $a \in A$ there is some $i \in \omega$ such that $\mathfrak{a}_A \models \varphi_i(\bar{a})$. Prove that for every $L$-structure $\mathfrak{b}$ elementarily equivalent to $\mathfrak{a}$, for every $b \in B$ there is an $i \in \omega$ such that $\mathfrak{b}_B \models \varphi_i(\bar{b})$.

3. a) Let $T$ be a complete theory in a countable language $L$ that has an infinite model. Prove that $T$ is $\omega$-categorical iff all models of $T$ realize precisely the same $n$-types for each $n \in \omega$.

b) Let $L$ be a countable language and let $\mathfrak{a}$ be an infinite, countable, saturated $L$-structure. Prove that there is a proper elementary extension $\mathfrak{b}$ of $\mathfrak{a}$ that is isomorphic to $\mathfrak{a}$.
4. a) Let $T$ be a theory in a language $L \supseteq \{S, 0\}$ that contains only finitely many non-logical symbols. Assume that every recursive relation is representable in $T$. Prove that $T$ is undecidable.

b) Let $L$ be a countable language and let $L' = L \cup \{c\}$, where $c$ is a constant symbol not in $L$. Let $\Sigma$ be a set of sentences of $L$, let $T = \text{Cn}_L(\Sigma)$ and let $T' = \text{Cn}_{L'}(\Sigma)$. Prove that $T$ is undecidable iff $T'$ is undecidable.

5. a) Let $E \subseteq \omega \times \omega$ be r.e. Assume that $E$ is an equivalence relation on $\omega$ and assume that $C \subseteq \omega$ is an r.e. set that contains exactly one element from each $E$-class. Prove that $E$ is recursive.

b) Let $A \subseteq \omega$ be non-empty. Carefully prove that $A$ is the domain of some partial recursive function iff $A$ is the range of some total recursive function.

6. a) Let $A$ be a non-empty, proper subset of $\omega$. Assume that $A$ is recursive. Prove that there are numbers $a \in A$ and $b \in (\omega \setminus A)$ such that $W_a = W_b$.

b) Let $X$ be a non-empty subset of $\omega$. Assume that $X$ is r.e. Let $I = \{e \in \omega : W_e = X\}$. Prove that every r.e. subset $A$ of $\omega$ is many-one reducible to $I$. 
1. a) Let $T$ be a theory of $L$ and let $\sigma$ be a sentence of $L$. Assume that for every model $\mathfrak{A}$ of $T$, $\mathfrak{A} \models \sigma$ iff $A$ is finite. Prove that there is some $n \in \omega$ such that every model of $T$ with at least $n$ elements is infinite.

b) Let $\mathfrak{A}$ be a proper elementary extension of $(\omega, <)$. Prove that there is an infinite sequence $\{a_n\}_{n \in \omega}$ of elements of $A$ such that $a_{n+1} < a_n$ holds for all $n \in \omega$.

2. a) Let $\mathfrak{A}$ be an infinite $L$-structure. Assume that for every formula $\varphi(x)$ of $L$, either $\varphi^2$ is finite or $(-\varphi)^2$ is finite. Prove that there is exactly one complete $1$-type $\Gamma(x)$ consistent with $T$ that can be realized by infinitely many elements in some model of $T$.

b) Let $T$ be a complete theory in a countable language $L$ and let $\Phi(x)$ be a type consistent with $T$. Assume that $\Phi$ is omitted in some model of $T$. Prove that there is another model of $T$ in which $\Phi$ is realized by infinitely many elements.

3. a) Let $T$ be a complete theory in the language $L = \{+, \cdot, <, S, 0\}$ such that $Q \subseteq T$ but $(\omega, +, \cdot, <, S, 0) \models T$. Prove that there is some formula $\varphi(x)$ of $L$ such that $T \models \exists x \varphi(x)$ but $T \models \neg \varphi(n)$ for every $n \in \omega$.

b) Let $\mathfrak{A}$ be the countable model of an $\omega$-categorical theory in a countable language $L$. Prove that $\mathfrak{A}$ has a non-trivial automorphism.
4. a) Prove that every infinite r.e. $A \subseteq \omega$ contains an infinite recursive subset.
   
   b) Let $R \subseteq \omega \times \omega$ be r.e. and satisfy the following conditions:
   
   $\bigcup_{k \in \omega} R_k = \omega$ and $R_k \cap R_l = \emptyset$ whenever $k \neq l$.
   
   Prove that $R$ is recursive. (Recall that $R_k = \{ l : R(k, l) \text{ holds} \}$).
   
5. a) Let $X \subseteq \omega$ be r.e. but not recursive. Let $\varphi(x)$ be a $\Sigma$-formula in the language $L = \{ +, \cdot, <, S, 0 \}$ that defines $X$ in $(\omega, +, \cdot, <, S, 0)$. Prove that there is some consistent theory $T \supseteq Q$ such that $T \vdash \varphi(\bar{n})$ for some $n \notin X$.

   b) Prove that there is a partial recursive function $f$ that cannot be extended to a total recursive function (i.e., there is no total recursive function $g$ such that $g(k) = f(k)$ whenever $f(k)$ is defined).

6. a) Prove that there is some $\epsilon \in \omega$ such that $\{\epsilon\}(2\epsilon) = 3\epsilon + 1$.

   b) Let $A = \{ [\sigma] : \sigma \text{ is a sentence of } L = \{ +, \cdot, <, S, 0 \} \text{ and } \models \sigma \}$. Prove that $A$ is a complete r.e. set.
1. a) Let $L$ be a countable language containing at least the binary relation symbol $E$, and let $T$ be a theory of $L$ so that $E^A$ is an equivalence relation on $A$ for every model $A$ of $T$. Assume that whenever $A$ is a model of $T$ and $B$ is an elementary extension of $A$ then every element of $(B - A)$ has its $E^B$-class contained in $(B - A)$. Prove that there is some integer $n$ such that in every model $A$ of $T$ every $E^A$-class has size $< n$.

b) Let $T$ be a consistent theory in the countable language $L$ and let $\exists \forall^\infty$ and $\forall^\exists$ be types consistent with $T$. Assume that for every model $A$ of $T$ we have $\exists^A = A - \forall^A$. Prove that there is some formula $\varphi(x)$ such that $\exists^A = \varphi^A$ for every model $A$ of $T$.

2. Let $T$ be a complete theory in a countable language $L$ and let $\exists^\infty(x)$ be a complete type of $T$. Assume that $T$ has models $A$ and $B$ so that $|\exists^A| = 1$ and $|\exists^B| = 2$.

a) Prove that $T$ has a model omitting $\exists^\infty$.

b) Prove that $T$ has a model $C$ so that $\exists^C$ is infinite.

3. a) Prove that $(\omega, +)$ has no proper elementary substructures.

b) Let $T$ be a complete $\omega$-categorical theory in a countable language. Prove that there is an integer $n$ such that for every formula $\varphi(x)$ and every model $A$ of $T$, if $\varphi^A$ is finite then $|\varphi^A| < n$. 
4. a) For any $R \subseteq \omega \times \omega$ we define $R_k = \{ l : R(k,l) \text{ holds} \}$. Assume that $R$ is r.e. and $\bigcup_{k \in \omega} R_k = \omega$. Prove that there is some recursive $S \subseteq R$ such that $\bigcup_{k \in \omega} S_k = \omega$ and further $S_k \cap S_{k+1} = \emptyset$ whenever $k \neq l$.

b) Let $A$, $B \subseteq \omega$ and assume that $B$ is r.e. but not recursive and that $B \leq_m A$. Prove that $A$ contains an infinite r.e. subset.

5. a) Prove that $\{ e : W_e \neq \omega \} \leq_m \{ e : W_e \text{ is finite} \}$.

b) Let $A_n$ be arbitrary subsets of $\omega$ for every $n$ in $\omega$. Prove that there is some $B \subseteq \omega$ such that $A_n \leq_T B$ for every $n$.

6. a) Prove that $REC = \{ e : W_e \text{ is recursive} \}$ is $\Sigma_3^0$.

b) Prove that $A \leq_T \{ \sigma^* : N \models \sigma \}$ for every arithmetic $A \subseteq \omega$, where $N$ is the standard model of arithmetic on the natural numbers.
1. a) Let $L$ be a language containing at least the binary relation symbol $E$. Let $A$ be an $L$-structure in which $E$ is interpreted as an equivalence relation on the universe. Assume that every element of every elementary extension of $A$ belongs to the $E$-class of some element of $A$. Prove that there are just finitely many $E$-classes in $A$.

b) Let $L$ and $L'$ be languages with $L \subseteq L'$. Let $T'$ be an $L'$-theory, and let $A$ be an $L$-structure. Assume that there is no model of $T'$ whose $L$-reduct is elementarily equivalent to $A$. Prove that there is some $L$-sentence $\sigma$ such that $A \not\models \sigma$ and $T' \not\models \neg \sigma$.

2. a) Let $T$ be a complete theory of a language $L$ and let $\bar{\varphi}(x)$ be an $L$-type. Assume that $\bar{\varphi}$ is realized by at most one element in every model of $T$. Prove that there is some formula $\varphi(x)$ such that $\bar{\varphi}^A = \varphi^A$ for every model $A$ of $T$.

b) Let $A$ be the countable model of an $\omega$-categorical theory in a countable language $L$. Let $X$ be a subset of $A$ fixed by all automorphisms of $A$ (that is, if $a \in X$ then $h(a) \in X$ for every automorphism $h$ of $A$). Prove that $X$ is definable in $A$ by some $L$-formula. (You may assume that if $(\bar{A},a) \equiv (\bar{A},b)$ then $(\bar{A},a) \not\equiv (\bar{A},b)$, and also the Ryll-Nardzewski characterization of $\omega$-categorical theories).
3. a) Prove that \( \text{Th}((\mathbb{Z}, +)) \) does not have a countable \( \omega \)-saturated model.

b) Let \( L \) be a countable language containing at least a binary relation symbol \( E \). Let \( T \) be an \( L \)-theory stating (among other things) that \( E \) is an equivalence relation on the universe. Assume that \( T \) has a model \( A \) with the property that every \( L \)-formula \( \varphi(x) \) satisfiable on \( A \) is satisfiable by some element of \( A \) from a finite \( E \)-class. Prove that \( T \) has a model in which all \( E \)-classes are finite.

4. a) Let \( R \) be a binary relation on \( \omega \) which is r.e. but not recursive.
Assume that \( R_k \cap R_{k+1} = \emptyset \) for all \( k \neq 1 \) (where \( R_k = \{ n : R(k,n) \text{ holds}\} \)).
Prove that \( \bigcup_{k \in \omega} R_k \) is not recursive.

b) Let \( A = \{ \sigma : q \vdash \sigma \} \) where \( Q \) is the theory of the language of arithmetic used in undecidability results. Prove that every r.e. set of natural numbers is many-one reducible to \( A \).

5. a) Assume \( X \subseteq \omega \) is such that \( \{ e : W_e = \omega \} \subseteq X \) and \( \{ e : W_e = \emptyset \} \cap X = \emptyset \).
Prove that \( X \) is not recursive.

b) Prove that \( B = \{ e : \{ e \}(2e) = 3 \} \) is a complete r.e. set.

6. a) Assume that \( B \subseteq \omega \) is infinite but contains no infinite r.e. subset.
Assume that \( A \) is r.e. and \( A \preceq^m B \). Prove that \( A \) is recursive.

b) Recall that \( \text{COF} = \{ e : (\omega - W_e) \text{ is finite} \} \). Prove that \( \text{COF} \) is r.e. in \( \emptyset' \).
1. a) Prove that \((\mathbb{Z}, <)\) has no proper elementary submodels.

b) Let \(T\) be a complete theory in a countable language \(L\) containing (at least) a binary relation symbol \(E\) such that in every model of \(T\), \(E\) is interpreted as an equivalence relation on the universe. Assume that in every \(\omega\)-saturated model of \(T\) there is exactly one infinite \(E\)-class. Prove that there is some integer \(n\) such that in every model of \(T\) every \(E\)-class with \(> n\) elements is infinite.

2. a) Let \(T\) be a consistent theory in a countable language \(L\). Assume that for all formulas \(\varphi(x)\) of \(L\) we have
\[
T \not\vdash \forall x \varphi(x) \iff T \not\vdash \varphi(c) \text{ for all constants } c \text{ of } L.
\]
Prove that \(T\) has a model \(A\) such that \(A = \{c^A : c \in L\}\).

b) Let \(A\) be any \(L\)-structure and assume that \(A\) realizes exactly three different complete types. Show that the same is true for every \(L\)-structure \(B\) elementarily equivalent to \(A\).

3. a) Let \(T\) be a complete theory in a countable language \(L\) and let \(A\) be a countable atomic model of \(T\). Assume that \(a\) and \(b\) are elements of \(A\) with the same complete type. Prove that \(A\) has an automorphism \(f\) such that \(f(a) = b\).

b) Let \(T\) be a complete theory in a countable language \(L\). Assume there are only finitely many complete types \(\bar{\Phi}(x)\) in a single variable \(x\) consistent with \(T\). Prove that there are only finitely many formulas \(\varphi(x)\) of \(L\) up to equivalence with respect to \(T\).
4. a) Let $A$ and $B$ be disjoint r.e. sets of natural numbers, and assume neither of them is recursive. Prove that $(A \cup B)$ is not recursive.

b) Prove that any theory $T$ with an r.e. set of axioms also has a recursive set of axioms.

5. a) Let $T$ be a theory in a countable language $L$ and assume that
\[{n \in \omega : T \text{ has a model of cardinality } n}\] is not recursive. Prove that $T$ is undecidable.

b) Let $T$ be a consistent recursively axiomatizable theory in the usual language for arithmetic on the natural numbers. Assume that $X$ is weakly representable in $T$ by $\varphi(x)$ and that $X$ is not recursive. Prove that there is some consistent recursively axiomatizable theory $T'$ containing $T$ such that $X$ is not weakly representable in $T'$ by $\varphi(x)$.

6. a) Prove that there are r.e. subsets $A$ and $B$ of $\omega$ which are disjoint but there is no recursive set $C$ with $A \subseteq C$ and $(B \cap C) = \emptyset$.

b) Prove that \[{e : W_e \text{ is infinite}}\] $\leq_m \{e : W_e = \omega \}$. 
[Hint: first define a partial recursive function $g(e,x)$ which converges iff \{e\}(y) converges for some $y > x$]
1. a) Let $A$ be an $L$-structure and let $\varphi(x)$ be a formula of $L$. Prove that $\varphi^A$ is finite iff $\varphi^B = \varphi^A$ for every elementary extension $B$ of $A$.

b) Let $T$ be a complete theory in a countable language $L$, let $A$ be an $\omega$-saturated model of $T$, and let $\bar{\Phi}(x)$ and $\bar{\Psi}(x)$ be $L$ types. Assume that $\bar{\Phi}^A = A - \bar{\Phi}^A$. Prove that there is some formula $\varphi(x)$ of $L$ such that $\varphi^A = \varphi^A$.

2. a) Let $T$ be a countable language whose non-logical symbols include the binary relation $\prec$. Let $T$ be a consistent theory of $L$ such that $\prec^A$ is a linear order of $A$ for every model $A$ of $T$. Assume that whenever $A$ is a model of $T$ there are $a, b$ in $A$ such that the $\prec^A$-interval between $a$ and $b$ is infinite. Prove that there is some formula $\varphi(x,y)$ of $L$ consistent with $T$ such that whenever $A$ is a model of $T$ and $A \models \varphi(a,b)$ then the $\prec^A$-interval between $a$ and $b$ is infinite.

b) Let $L$ and $L'$ be languages with $L \subseteq L'$, let $T_1'$ and $T_2'$ be theories of $L'$ which contain precisely the same sentences of $L$, and let $T$ be a theory of $L$. Prove that some model of $T$ can be expanded to a model of $T_1'$ iff some model of $T$ can be expanded to a model of $T_2'$.

3. a) Let $A$ be any $L$-structure, let $L' = L(A)$ and let $T' = \text{Th}(A)$. Let $B'$ be an $L'$-structure which is a model of $T'$. Assume that $B'$ is an atomic model of $T'$. Prove that $B$, the $L$-reduct of $B'$, is isomorphic to $A$.

b) Let $T$ be a complete $\omega$-categorical theory in a countable language $L$. Prove that there is some integer $k$ such that for every formula $\varphi(x)$ of $L$ and every model $A$ of $T$, if $|\varphi^A| > k$ then $\varphi^A$ is infinite.
4. a) Assume that $R \subseteq \omega \times \omega$ is r.e. and defines a strict linear order on $\omega$ with no last element (so $R(k,k)$ fails for all $k$). Prove that there is a strictly increasing recursive function $f$ such that $R(f(k), f(k+1))$ holds for all $k$.

b) Let the non-logical symbols of $L$ be $\{+,-,\cdot,0\}$ and let $\mathbb{N}$ be the standard $L$-structure for arithmetic on the natural numbers. Prove that there is no listing $\{\varphi_n(x): n \in \omega\}$ of all the formulas of $L$ with $x$ free such that $X = \{n: \mathbb{N} \models \varphi_n(\bar{n})\}$ is recursive.

5. a) Let $L$ have as its only non-logical symbol the binary relation $E$ and let $T_0$ be the $L$-theory asserting that $E$ is an equivalence relation on the universe with infinitely many classes. Prove that there is a complete $L$-theory $T$ which extends $T_0$ and is undecidable.

b) Let $A$ be a non-empty r.e. subset of $\omega$ and define $I = \{e: A = W_e\}$. Prove that every r.e. set $B$ is many-one reducible to $I$.

6. a) Let $L$ be a language with finitely many non-logical symbols and let $L' = L \cup \{c\}$ where $c$ is an individual constant symbol not in $L$. Let $A'$ be a strongly undecidable $L'$-structure and let $A$ be its reduct to $L$. Prove that $\text{Th}(A)$ is an undecidable $L$-theory.

b) Let $\text{REC} = \{e: W_e \text{ is recursive}\}$. Prove that $\text{REC}$ is r.e. in $\emptyset''$. 
1. a) Let \( L \) be a language whose non-logical symbols include the binary relation \( E \). Let \( T \) be a theory of \( L \) such that \( E^A \) is an equivalence relation on \( A \) for every model \( A \) of \( T \). Assume that in every model \( A \) of \( T \) there is exactly one infinite \( E^A \)-class. Prove that there is some \( n \) in \( \omega \) such that in every model \( A \) of \( T \) all finite \( E^A \)-classes have at most \( n \) elements.

b) Let \( T \) be a complete theory of some language \( L \) and let \( \bar{\Phi} (x) \) be an \( L \)-type consistent with \( T \). Assume that \( \bar{\Phi} \) is omitted on some model of \( T \). Prove that \( \bar{\Phi} \) is realized in some model of \( T \) by at least two different elements.

2. a) Let \( T \) be a complete theory in a countable language \( L \) and let \( A \) be the prime model of \( T \). Let \( \bar{\Phi} (x) \) be any \( L \)-type. Prove that there is some \( L \)-type \( \bar{\Psi} (x) \) such that \( \bar{\Psi}^A = A - \bar{\Phi}^A \).

b) Let \( L \) be a countable language and let \( L' = L \cup \{ c_1, \ldots, c_k \} \) where \( c_1, \ldots, c_k \) are individual constants not in \( L \). Let \( T \) and \( T' \) be complete theories of \( L \) and \( L' \) respectively and assume \( T \subseteq T' \). Prove that \( T \) has a countable universal model iff \( T' \) has a countable universal model.

3. a) Let \( L \) be a countable language. An \( L \)-structure \( A \) is said to be locally finite iff every element of \( A \) belongs to a finite \( L \)-definable subset of \( A \). Let \( T \) be a complete \( L \)-theory and assume that no model of \( T \) is locally finite. Prove that there is some \( L \)-formula \( \varphi (x) \) consistent with \( T \) such that for every \( L \)-formula \( \psi(x) \) and every model \( A \) of \( T \), \( \varphi^A \cup \psi^A \) is infinite provided it is not empty.
b) Let T be a complete theory in a countable language L. Let A be a countable model of T which is not prime and let \( \Phi(x) \) be a type omitted on A. Prove that there is some countable model of T which also omits \( \Phi \) but is not isomorphic to A.

[Warning: You cannot assume that T has a prime model.]

4. a) Assume that A and B are r.e. subsets of \( \omega \) such that \( A \cup B \) is recursive. Prove that there are recursive sets \( A' \subseteq A \) and \( B' \subseteq B \) such that \( A \cup B = A' \cup B' \).

b) Recall that if \( \psi(x) \) is a \( \Sigma \)-formula (in the language for arithmetic on the natural numbers) and if \( Q \vdash \exists x \psi(x) \) then \( Q \vdash \psi(\bar{n}) \) for some n in \( \omega \). Prove that there is no total recursive function f such that whenever \( \psi(x) \) is a \( \Sigma \)-formula and \( Q \vdash \exists x \psi(x) \) then \( Q \vdash \psi(\overline{f(k)}) \) where \( k = \gamma \psi^\neg \).

[Hint: Let \( \psi(x,y) \) be a \( \Sigma \)-formula representing in Q the relation "x is the Gödel number of a proof from Q of the sentence whose Gödel number is y" and consider the formulas \( \varphi_1(x) = \psi(x,\overline{1}) \).]

5. a) Given a language \( L_1 \) let \( L_2 = L_1 \cup \{ c \} \) where c is an individual constant not in \( L_1 \). Let \( T_2 \) be a finitely axiomatizable essentially undecidable theory of \( L_2 \) and let \( T_1 = T_2 \cap S_{\omega} \). Prove that \( T_1 \) is also essentially undecidable.

b) Prove that \( \{ e : W_e \not= \omega \} \preceq_m \{ e : W_e \text{ is finite} \} \).

[Hint: First define a partial recursive function \( f(e,x) \) which converges iff \( \{ e \} (y) \) converges for all \( y < x \).]

6. a) Let A and B be subsets of \( \omega \). Prove that B is A-r.e. iff \( B \preceq_m A' \) where \( A' \) is the jump of A.

b) Let \( C = \{ \overline{\sigma^\neg} : N \models \sigma \} \) where N is the standard model of arithmetic on the natural numbers. Prove that \( A \preceq_T C \) for all arithmetic sets A, and use this to conclude that C is not arithmetic.
1. a) Given a theory $T$ and a sentence $\vartheta$ of $L$, assume that for every model $A$ of $T$, $A \models \vartheta$ iff $A$ is finite. Prove that there is some $n \in \omega$ such that for every model $A$ of $T$, $A \models \vartheta$ iff $A$ has at most $n$ elements.

b) Let $A$ and $B$ be $L$-structures and assume that $B$ is a proper elementary extension of $A$. Assume further that there is an $L$-formula $\varphi(x,y)$ such that $A = \{ b \in B : B_B \models \varphi(b, b_0) \}$ for some $b_0$ in $B$. Prove that $b_0 \notin A$.

2. a) Let $T = \text{Th}((\mathbb{Q},+,\cdot,<,0,1))$. Prove that $T$ does not have a countable saturated model.

b) Let $T$ be a complete $L$-theory, let $L'$ be a language containing $L$ and let $T'$ be an $L'$-theory containing $T$. Assume that $A$ is a model of $T$ which has an elementary extension which can be expanded to an $L'$-structure which is a model of $T'$. Prove that every model $B$ of $T$ has an elementary extension which can be expanded to a model of $T'$.

3. Let $L$ be a countable language containing (at least) a binary relation symbol $\prec$ and individual constants $c_n$ for all $n \in \omega$. Let $T$ be a complete theory of $L$ containing (at least) the axioms that $\prec$ is a linear order of the universe and $c_n \prec c_{n+1}$ for all $n \in \omega$. Call a model $A$ of $T$ standard if for every $a \in A$ there is some $n \in \omega$ such that $A_A \not\models \bar{a} \prec c_n$. Let $A^*$ be an $\omega$-saturated model of $T$.

a) Prove that if $A^*$ is standard then there is some $n \in \omega$ such that $A^* \models \forall x (x \preceq c_n)$.

b) Assume that for every $L$-formula $\varphi(x)$ such that $A^* \models \exists x \varphi(x)$ there is some $n \in \omega$ such that $A^* \models \exists x [ \varphi(x) \land x \preceq c_n ]$. Prove that $T$ has a standard model.
4. Let $T$ be a recursively axiomatized extension of the theory $Q$ which is true on $\mathbb{N} = (\omega, +, *, <, 0, s)$. Let $R \subseteq \omega \times \omega$ be representable in $T$ by the $\Sigma_1$-formula $\varphi(x, y)$. Let $X = \{k : \exists l \ R(k, l) \ \text{holds}\}$.

a) Show $X$ is weakly representable in $T$ by $\exists y \ \varphi(x, y)$.

b) Assume $X$ is not recursive. Prove that there is some $k \in \omega$ such that $T \vdash \neg \varphi(k, 1)$ for all $l \in \omega$ but $T \not\vdash \forall y \neg \varphi(k, y)$.

5. a) Let $\mathcal{F}$ be a set of partial recursive functions of one argument, and let $I = \{e : (e) \in \mathcal{F}\}$. Prove that $I \notin \mathfrak{m}(\omega - I)$.

b) Let $A$ and $B$ be subsets of $\omega$. Assume $B$ is r.e. but not recursive and that $B \leq_m A$. Prove that $A$ contains an infinite r.e. subset.

6. a) Let $L_0$ be the language with no non-logical symbols.

i) Show that there is a theory $T_0$ of $L_0$ which is undecidable.

ii) Can there be an undecidable $L_0$-theory $T_0$ which has only finite models? Explain.

b) Let $X$ be an r.e. subset of $\omega$. Let $I = \{e : W_e = X\}$. Prove that $I$ is $\Pi_2$. 