Qualifying Exam AMSC/CMSC 666

1. Let \( p_3(x) \) be the second-degree polynomial which interpolates the given values \( y_j = f(x_j) \) at the three distinct nodes, \( x_1 < x_2 < x_3 \).

(a) We are now given the additional value of the derivative, \( y'_3 = f'(x_3) \). Let \( p_4(x) \) be the polynomial which interpolates the three values, \( p_4(x_j) = y_j, \ j = 1, 2, 3, \) and the additional derivative \( p'_4(x_3) = y'_3 \). Then \( p_4(x) \) has the form \( p_4(x) = p_3(x) + q(x) \). Write an explicit expression for \( q(x) \).

(b) Give a formula for the interpolation error \( f(x) - p_4(x) \) using the derivatives of \( f \).

(c) Suppose we change the value at node \( x_1 \) from \( y_1 = y_1 + \varepsilon \), while the data at the other nodes, \( y_2, y_3 \) and \( y'_3 \) remains unchanged. Express the corresponding interpolating polynomial \( \tilde{p}_4(x) \) in the form \( \tilde{p}_4(x) = p_4(x) + \varepsilon \tau(x) \) for an appropriate \( \tau(x) \).

2. We want to approximate the integral \( I(f) = \int_0^1 f(x) \sqrt{x} \, dx \) by a quadrature formula \( Q_n(f) := \sum_{j=1}^n w_j f(x_j) \) at the nodes \( \{x_j\}_{j=1}^n \) using the weights \( \{w_j\} \).

(a) Assume that there exists a polynomial \( p \) for which the quadrature \( Q_n \) is exact: \( Q_n(p) = I(p) \). Prove that if \( w_j \geq 0 \) then

\[
|I(f) - Q_n(f)| \leq \frac{4}{3} \max_{x \in [0,1]} |f(x) - p(x)|.
\]

Hint: verify that \( \sum_{j=1}^n w_j = \int_0^1 \sqrt{x} \, dx = \frac{2}{3} \) and that \( I(f) - Q_n(f) = I(f - p) + Q_n(p - f) \).

(b) Assume that \( f \) is continuous, and let \( Q_n(f) \) be the Gaussian quadrature with \( n \) nodes. Use (a) to show that

\[
\lim_{n \to \infty} Q_n(f) = I(f).
\]  \hspace{1cm} (1)

State which properties of Gaussian quadrature you use in your argument (you don’t have to prove them).

(c) Does (1) hold for a Newton-Cotes type quadrature, \( Q_n(f) \), which is based on the \( n \) equi-spaced nodes, \( x_j = j/n, \ j = 1, 2, \ldots, n \) ? If so then prove it; if not then explain what is the relevant difference to Gaussian quadrature.

3. We want to solve the nonlinear system \( f_1(x, y) = 0 \) and \( f_2(x, y) = 0 \) where \( f_1, f_2 \in C^2(D) \) for an open set \( D \subset \mathbb{R}^2 \). We use the iterative method

\[
x_{k+1} := x_k - \frac{1}{2} \frac{\partial f_1}{\partial x}(x_k, y_k) \quad y_{k+1} := y_k - \frac{1}{2} \frac{\partial f_2}{\partial y}(x_k, y_k).
\]

Assume that \( \left| \frac{\partial}{\partial x} f_1(x, y) \right| > \left| \frac{\partial}{\partial y} f_1(x, y) \right| \) and \( \left| \frac{\partial}{\partial y} f_2(x, y) \right| > \left| \frac{\partial}{\partial x} f_2(x, y) \right| \) for all \( (x, y) \in D \). Show that this iteration is locally convergent for \( (x_0, y_0) \in D \) with \( f_1(x_0, y_0) = 0 \) and \( f_2(x_0, y_0) = 0 \). Hint. Consider the Jacobian of the iteration function.
Let \( u(x) = x^\alpha \) with \( \frac{1}{2} < \alpha < 1 \) and \( x \in \Omega = (0, 1) \). Consider the 2-point boundary value problem
\[-u'' = f(x) \text{ with } u(0) = 0 \text{ and } u(1) = 1.\]

(a) Define \( H^1(\Omega) \) and \( H^{-1}(\Omega) \). Show that \( u \in H^1(\Omega) \), \( f \in H^{-1}(\Omega) \), and \( f \not\in L^1(\Omega) \).

(b) Let \( T = \{x_i\}_{i=0}^N \) be a partition of \( \Omega \) with \( x_0 = 0 \) and \( x_N = 1 \), let \( V_T \) be the finite element space of continuous piecewise linear functions over \( T \), and let \( U \in V_T \) be the Galerkin solution. Write the equation satisfied by \( U \), including boundary conditions for both \( U \) and the test function, and show that
\[\|u'' - U''\|_{L^2(\Omega)} = \inf_{V \in W_T} \|u'' - V''\|_{L^2(\Omega)},\]
where \( W_T \) is a subset of \( V_T \) with suitable boundary conditions. Make these boundary conditions explicit.

(c) Suppose that \( T \) is a quasi-uniform mesh. State the usual error bound for \( \|u'' - U''\|_{L^2(\Omega)} \), and comment on whether this bound is applicable for the example \( u(x) \) of this problem.

Consider a linear system of equations \( Ax = b \) where \( A \) is a symmetric positive-definite matrix of order \( n \). The conjugate gradient method (CG) is an iterative solution algorithm for such problems that computes a sequence of approximate solutions \( \{x_j\} \) satisfying
\[\|x - x_j\|_A = \min_{x \in \Pi_j} \|p_j(A)(x - x_0)\|_A,\]
where \( \|\cdot\|_A \) is the inner product \( (Au, v)^{1/2} \) and \( \Pi_j \) is the set of polynomials of degree \( j \) with value 1 at the origin.

(a) Show that
\[\frac{\|x - x_j\|_A}{\|x - x_0\|_A} \leq \min_{x \in \Pi_j} \frac{\max_{\lambda \in \sigma(A)} |p_j(\lambda)|}{\max_{\lambda \in \sigma(A)} |p_j(\lambda)|},\]
where \( \sigma(A) \) is the set of eigenvalues of \( A \).

(b) Suppose the eigenvalues of \( A \) lie in two disjoint intervals \( [a, b] \cup [c, d] \) of equal length. It can be shown that there is a unique quadratic polynomial \( p_2(x) \) that satisfies \( p_2(0) = 1 \), \( p_2(b) = -p_2(a) \), and
\[0 < \frac{1 - p_2(a)}{\alpha} = 1 - p_2(d) \leq 1 - p_2(\lambda) \leq \frac{1 - p_2(b)}{\beta} = 1 - p_2(c).\]
for \( \lambda \in [a, b] \cup [c, d] \). Show that the CG iterate at step \( 2k \) satisfies
\[\frac{\|x - x_{2k}\|_A}{\|x - x_0\|_A} \leq \max_{\lambda \in \sigma(A)} \frac{T_k \left( \frac{\beta + \alpha - 2(1 - p_2(\lambda))}{\beta - \alpha} \right)}{T_k \left( \frac{\beta + \alpha}{\beta - \alpha} \right)},\]
where \( T_k(x) \) is the Chebyshev polynomial of degree \( k \).

(c) Use this result to show that for the relative error of the \( 2k^{th} \) iterate to be smaller than a tolerance \( \tau \), \( k \) should approximately satisfy
\[k \geq \frac{1}{2} \sqrt{\frac{\beta}{\alpha}} \ln \frac{2}{\tau}.\]
Comment. The standard analysis of CG gives
\[ 2k \geq \frac{1}{\alpha} \sqrt{d \ln \frac{a}{r}}. \]
where $\frac{a}{r}$ is the condition number of $A$. If $d \gg a$, then the bound from part (c) may show that a smaller number of iterations will be required.

6. Given a matrix $S = [s_1, \ldots, s_N] \in \mathbb{R}^{M \times N}$ of rank $K \leq N$ and $M \geq N$, the **proper orthogonal decomposition (POD)** is a method to approximate $S$ with $d < K$ linearly independent vectors in $\text{span}\{s_i\}_{i=1}^N$. Let $S = U\Sigma V^T$ be the singular value decomposition of $S$, i.e.
\[ U = [u_1, \ldots, u_M] \in \mathbb{R}^{M \times M}, \quad V = [v_1, \ldots, v_N] \in \mathbb{R}^{N \times N} \]
are orthogonal matrices and $\Sigma = \begin{bmatrix} \sigma_1 & \vdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_N \end{bmatrix} \in \mathbb{R}^{M \times N}$, with diagonal matrix $D = \text{diag} (\sigma_1, \ldots, \sigma_N)$ containing the singular values of $S$:
\[ \sigma_1 \geq \cdots \geq \sigma_K > \sigma_{K+1} = \cdots \sigma_N = 0. \]

(a) Show that $\text{span}\{s_j\}_{j=1}^N = \text{span}\{u_j\}_{j=1}^K$.

(b) Let $A_i = u_i v_i^T$ be a rank-1 matrix for $1 \leq i \leq N$. Show the orthogonal decomposition $S = \sum_{i=1}^K \sigma_i A_i$ in the Frobenius inner product
\[ (A, B)_F = \sum_{i=1}^M \sum_{j=1}^N a_{ij} b_{ij}, \]

i.e. show that $(A_i, A_j)_F = \delta_{ij}$ where $\delta_{ij}$ is the Kronecker delta.

(c) For $d < K$ we may approximate $S$ with $\sum_{i=1}^d \sigma_i A_i$; this is the POD of $S$. Show the error formula
\[ \left\| S - \sum_{i=1}^d \sigma_i A_i \right\|_F^2 = \sum_{i=d+1}^K \sigma_i^2, \]
where $\| \cdot \|_F$ is the norm subordinate to the Frobenius inner product. This expression could be used to find the number $d$ so that the error $\sum_{i=d+1}^K \sigma_i^2$ is within a given tolerance.