Qualifying Exam AMSC/CMSC 666/667

1. Find the polynomial $p$ of the form $p(x) = a - bx^2$ that provides the best approximation for the function $f(x) = \sqrt{1 - x^2}$ on the interval $[-1, 1]$ in the $L_\infty$ norm, i.e., $p$ must be such that the norm of the error $\|f - p\|_\infty = \max_{[-1,1]} |f(x) - p(x)|$ is as small as possible. What is $\|f - p\|_\infty$ for this $p$?

2. Consider the composite trapezoidal rule for evaluating the integral $I(f) = \int_a^b f(x)dx$, where $a$ and $b$ are finite and $f(x)$ is $m$ times continuously differentiable where $m \geq 4$. Show that one step of Richardson extrapolation applied to the composite trapezoidal rule leads to the composite Simpson rule. Justify the application of the Richardson extrapolation. (The other name for this is the Romberg integration.)

3. The trust region methods for finding minimum of a function over a domain $\mathbb{R}^n$ use the following strategy. Given the $k^{th}$ iterate, $x_k$, one solves a constrained minimization problem of the form

$$
\text{argmin}\{f_k + g_k^T p + \frac{1}{2} p^T B_k p : |p| \leq \Delta_k\},
$$

where $f_k = f(x_k)$, $g_k = \nabla f(x_k)$, $B_k$ is an approximation to the Hessian matrix $\nabla \nabla f(x_k)$, $|\cdot|$ denotes the Euclidean norm, and $\Delta_k$ is the radius of the trust region for the $k^{th}$ step. The Cauchy point $p^c$ is the minimizer of the above problem when $p$ is restricted to lie along the steepest descent direction $-g_k$. Find the Cauchy point $p^c$.

*Hint:* Consider two cases: $g_k^T B_k g_k \leq 0$ and $g_k^T B_k g_k > 0$.

4. Consider the two-step method

$$
y_{n+1} = -9y_n + 10y_{n-1} + \frac{h}{2}[13f(x_n, y_n) + 9f(x_{n-1}, y_{n-1})]
$$

with the uniform time step $h = x_{n+1} - x_n$ for all $n$ for the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$. This method can be initialized e.g. by making the Forward Euler first step.

(a) Show the method is consistent but not stable. What is its order?

(b) Suppose we use the method to solve

$$
y' = 0, \quad y(0) = c,
$$

with $y_0 = c$ and an error is made in the first step which yields $y_1 = c + \varepsilon$ where $\varepsilon$ is small and positive. For every $n > 1$ give an explicit formula for $y_n$. Describe how the error behaves as $n \to \infty$.

5. Consider the boundary value problem

$$
-u'' + (1 + x)u = x^2, \quad u'(0) = 1, \quad u(1) = 1
$$

(a) Discretize the problem. Take a uniform partition of $[0, 1]$

$$
x_i = ih, \quad i = 0, 1, 2, \ldots, n, \quad h = 1/n.
$$

Use the three point difference formula for $u''$ and any suitable difference formula for the boundary condition at $x = 0$. Write the result as a matrix-vector equation.
(b) Prove that the equation found in (b) has a unique solution.

6. Methods such as steepest descent or conjugate gradients are based on the fact that solving $Ax = b$, $A \in \mathbb{R}^{n \times n}$ symmetric, positive definite is equivalent to minimizing

$$\phi(x) = \frac{1}{2} x^T A x - x^T b$$

The idea is: given an approximation $x_{k-1}$ and a search direction $u_k$, we define

$$x_k = x_{k-1} + \alpha_k u_k$$

where $\alpha_k$ is chosen so that

$$\phi(x_k) = \min_{\alpha} \phi(x_{k-1} + \alpha u_k).$$

(a) Given $x_{k-1}$ and $u_k$ compute $\alpha_k$.

(b) If the search directions $\{u_1, \ldots, u_n\}$ are orthogonal (in the usual sense) will the solution $x$ be obtained after $n$ steps (in exact arithmetic)? Explain.

*Hint.* Think about the 2D case and the search directions parallel to the coordinate axes.