1. Given a differentiable function $f(x)$ over $[0, 4]$ let $S(x)$ be the cubic spline interpolant of its values at the points $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3,$ and $x_4 = 4,$ that satisfies the boundary conditions $S'(0) = f'(0)$ and $S'(4) = f'(4).$ Find the expression for $S(x)$ on the interval $[0, 1]$ when $f(x) = \cos\left(\frac{\pi x}{2}\right)$ (see Figure 1).

*Hint: Use symmetries of $\cos\left(\frac{\pi x}{2}\right)$ to reduce the size of the problem.*

![Figure 1: Graph of $f(x) = \cos\left(\frac{\pi x}{2}\right)$ over $[0, 4]$.](image)

2. Consider an adaptive quadrature method based on the trapezoidal rule where each interval is divided into 3 equal subintervals whenever the error estimate exceeds the tolerance. Derive and justify an error estimate for this method over the interval $[a, b]$ of the form

$$E = \alpha|T(a, c) + T(c, d) + T(d, b) - T(a, b)|,$$

where $c = a + \frac{1}{3}(b - a), d = a + \frac{2}{3}(b - a),$

$$T(x, y) = \frac{1}{2}(y - x)[f(x) + f(y)],$$

and $\alpha$ is to be found.

3. (a) Suppose you need to minimize $f(x) = \frac{1}{2}x^T A x + b^T x + c, x \in \mathbb{R}^n,$ where $A$ is symmetric positive definite. Let $x_0$ be the starting point. Perform one step of the steepest descent method with exact line search and find the expression for $x_1.$

(b) Let $f(x), x \in \mathbb{R}^n,$ be a smooth arbitrary function. The BFGS method is a quasi-Newton method whose update formula for the Hessian approximate is given by

$$B_{k+1} = B_k - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k} + \frac{y_k y_k^T}{y_k^T s_k},$$

where $s_k := x_{k+1} - x_k, y_k := \nabla f_{k+1} - \nabla f_k.$

Let $x_0$ be the starting point and let the initial approximation for the Hessian be the identity matrix. What condition involving $s_k$ and $y_k$ do you need to verify at every BFGS step to ensure that $B_{k+1}$ is symmetric positive definite if $B_k$ is? Explain your answer.
4. Consider the $n \times n$ matrix

$$A = \begin{pmatrix}
2 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & \ddots & \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & -1 & 2 & -1 \\
0 & \cdots & 0 & -1 & 1
\end{pmatrix}.$$ 

(a) Show that $A$ is an irreducible matrix. (You may use a theorem.)

(b) Show that $A$ is an irreducibly diagonally dominant matrix.

(c) Show that $A$ is invertible without using the theorem that “Every irreducibly diagonally dominant matrix is invertible.”

(d) Show that $A$ is a symmetric positive definite matrix without using the theorem that “Every symmetric, irreducibly diagonally dominant matrix that has positive diagonal entries is positive definite.”

5. Consider the (unshifted) $QR$-method for finding the eigenvalues of an invertible matrix $A$.

(a) Give the algorithm that generates the sequence of matrices $\{A_m\}_{m=0}^{\infty}$ with $A_0 = A$.

(b) Show that each of the matrices $A_m$ are orthogonally similar to $A$.

(c) Compute $A_1$ when

$$A = \begin{pmatrix}
3 & 4 \\
4 & 7
\end{pmatrix}.$$ 

(d) The sequence $\{A_m\}$ generated by this algorithm for the $A$ given above has a limit $A_{\infty}$. (You do not have to prove that this limit exists.) What are the diagonal entries of $A_{\infty}$? Give your reasoning.

6. Consider initial-value problems over $\mathbb{R}$ in the form

$$x' = f(x), \quad x(0) = x_0 \in \mathbb{R},$$

where $f : \mathbb{R} \to \mathbb{R}$ is smooth with derivatives of all orders that are bounded over $\mathbb{R}$. One-step methods with a uniform time step $h > 0$ have the form

$$x_{m+1} = x_m + h \Phi(x_m, h) \quad \text{for } m = 0, 1, \cdots.$$ 

(2)

Recall that if there exists $p > 0$ such that for every solution $x(t)$ of (1) we have the bound

$$x(t+h) = x(t) + h \Phi(x(t), h) + O(h^{p+1}) \quad \text{as } h \to 0,$$

(3)

and $\Phi(x, h)$ satisfies a Lipschitz bound then the method is $p^{th}$-order with a convergence estimate $\|x(t) - x_m\| = O(h^p)$ as $h \to 0$ where $m$ satisfies $t = mh$.

Consider the numerical method

$$k_0 = hf(x_m), \quad y_1 = x_m + \frac{2}{3}k_0,$$

$$k_1 = hf(y_1), \quad y_2 = x_m + \frac{2}{3}[k_0 + k_1],$$

$$k_2 = hf(y_2), \quad x_{m+1} = x_m + \frac{1}{4}k_0 + \frac{4}{5}k_2.$$ 

(a) This method can be put into the form (2). Express $\Phi(x, h)$ in terms of $f$, $x$, and $h$.

(b) Determine the order of this method by establishing (3) for the optimal integer $p$. 