Instructions. Answer each the following six questions. Use a different answer sheet (or a different set of sheets) for each question. Write the problem number and your code number (not your name) on the top of each answer sheet. Keep scratch work on separate sheets.

Your work on each question will be assigned a grade from 0 to 10. Some problems have multiple parts or ask you to do more than one task. Be sure to go on to subsequent parts even if there is some part that you cannot do. Parts of a question need not have the same weight.

Carefully show all your steps, justify all your assertions, state precisely any definitions and theorems that you use, and explain your arguments in complete English sentences. Cross out any material that is not to be graded.

(1) Find an integral for the system

\[ \dot{x} = y \quad \dot{y} = -x^3 + 3x^2 - 2x \]

and sketch its phase portrait. What are equilibrium points? Which are Lyapunov stable, Lyapunov asymptotically stable?

(2) Let

\[ A = \begin{bmatrix} -2 & -1 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad g(t) = \begin{pmatrix} -\sin t \\ -\cos t \end{pmatrix}. \]

(a) Find the fundamental matrix solution to the equation \( \dot{x} = Ax \);
(b) Find the solution to the initial value problem

\[ \dot{x} = Ax + g(t) \quad \text{and} \quad x(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \]

(3) Let \((X(x, y), Y(x, y))\) be bounded vector field on the plane. Assume moreover that \((X, Y)\) is continuous on \(\mathbb{R}^2 - \Gamma\) where \(\Gamma\) is a segment \(\{0 \leq x \leq 1, y = 0\}\). Let \((x(t), y(t))\) be a solution to the equation

\[ \dot{x} = X, \quad \dot{y} = Y \]

which is defined on the interval \([0, \bar{t}]\) and which can not be continued beyond \(\bar{t}\). Show that \(\lim_{t \to \bar{t}} x(t)\) exists and \(\lim_{t \to \bar{t}} y(t) = 0\).

(4) Find all bounded odd solutions to the equation

\[ x''' + 2x'' = 8x. \]
(5) Consider the equation

\[ \dot{x} = -\sin(x) + 3y - y^5 \quad \dot{y} = e^x + y - \cos y. \]

Show that if \( a \) is a sufficiently small positive number then for each \( x_0 \) such that \( 0 < x_0 < a \) there exists a unique solution \((x(t), y(t))\) defined for all positive times and having properties (A)–(C) below.

(A) \( \lim_{t \to +\infty} x(t) = \lim_{t \to +\infty} y(t) = 0 \)
(B) \( x(0) = x_0, \)
(C) \( x(t) \) is strictly decreasing on \([0, \infty)\).

(6) Consider an equation \( \dot{x} = k(t) - x^2 \) where \( k \) is a continuous function such that, for all \( t \), \( 1 \leq k(t) \leq 2 \).

(a) Show that if \( x(0) \geq 0 \) then the solution is defined for all positive times.
(b) Show that if \( x_1(t) \) and \( x_2(t) \) are two solutions such that \( x_1(0) \geq 0 \) and \( x_2(0) \geq 0 \) then \( \lim_{t \to +\infty} [x_1(t) - x_2(t)] = 0. \)