Instructions. Answer all six questions. Your work on each question will be assigned a grade from 0 to 10. Your grade will be based on the work that is shown as well as your answer. Carefully show all your steps, justify all your assertions, state precisely any definitions and theorems that you use, and explain your arguments in complete English sentences. Cross out any material that is not to be graded. Some problems may have multiple parts or ask you to do more than one thing. Be sure to go on to subsequent parts even if there is some part you cannot do. If you are asked to prove a result and then apply it to a given situation you may receive partial credit for the correct application.

1. Consider a linear system $X' = AX$, where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

a) Find matrix exponential $e^{At}$.
b) Find all $X_0$ such that solutions $X(t)$ satisfying $X(0) = X_0$ are unbounded for $t \geq 0$.
c) Find all $X_0$ such that solutions $X(t)$ satisfying $X(0) = X_0$ are unbounded for $t \leq 0$.

2. Consider an ODE $\dot{x} = f(t, x)$, where $f(t, x)$ is continuous for all $(t, x) \in \mathbb{R}^2$.
Suppose an initial value problem $\dot{x} = f(t, x), \ x(t_0) = x_0$ has two distinct solutions. Prove that it has infinitely many distinct solutions.

3. Consider a linear ODE $\ddot{y} + \alpha(t) y = 0$, where $\alpha(t)$ is continuous for all $t \in \mathbb{R}$.
Let $\phi_1(t), \phi_2(t)$ be two solutions which are not multiples of each other. Suppose

$$\lim_{t \to \infty} (|\phi_1(t)| + |\phi_1(t)|) = 0$$

Prove that

$$\lim_{t \to \infty} (|\phi_2(t)| + |\phi_2(t)|) = \infty$$

4. Let $p$ be a fixed point of a $C^1$ system $\dot{x} = f(x), x \in \mathbb{R}^n$.
a) Give the definitions of positive stability and positive asymptotic stability of $p$. Here positive means for $t \geq 0$. Then give similar definitions of negative stability and negative asymptotic stability for $t \leq 0$.
b) Prove that $p$ cannot be simultaneously positively asymptotically stable and negatively stable.
c) Give an example of a system with an isolated fixed point which is simultaneously positively stable and negatively stable.
5. Consider an ODE

\[ \ddot{x} + h(x, \dot{x})\dot{x} + g(x) = 0 \]

where \( h(u, v) \) and \( g(u) \) are continuously differentiable for all \( (u, v) \).
Suppose \( ug(u) > 0 \) for \( u \neq 0 \), \( \int_{0}^{\infty} g(u) du = \infty \), \( \int_{0}^{-\infty} g(u) du = \infty \), and \( h(u, v) > 0 \).
Let \( X = f(X) \) be an equivalent first order system in \( \mathbb{R}^2 \).
Prove that for any initial condition \( X(0) \in \mathbb{R}^2 \) its omega limit set coincides with the origin.

6. Consider the system

\[ \begin{align*}
\dot{x} &= -y - x(\mu - (x^2 + y^2 - 1)^2) \\
\dot{y} &= x - y(\mu - (x^2 + y^2 - 1)^2)
\end{align*} \]

a) Find all values of \(-\infty < \mu < \infty\) corresponding to bifurcations of periodic orbits and bifurcations of fixed points.
b) Sketch phase portraits for bifurcation values of \( \mu \) and for sample values of \( \mu \) representing all different non-bifurcation types of dynamics.
c) Determine all types of bifurcations.