Instructions. Answer all six questions. Your work on each question will be assigned a grade from 0 to 10. Your grade will be based on the work that is shown as well as your answer. Carefully show all your steps, justify all your assertions, state precisely any definitions and theorems that you use, and explain your arguments in complete English sentences. Cross out any material that is not to be graded. Some problems may have multiple parts or ask you to do more than one thing. Be sure to go on to subsequent parts even if there is some part you cannot do. If you are asked to prove a result and then apply it to a given situation you may receive partial credit for the correct application.

1. Let $A$ and $B$ be real, constant $n \times n$ matrices. Let $\phi(t, \xi)$ be the solution of the initial value problem $\dot{x} = Ax$, $x(0) = \xi$, and let $\psi(t, \eta)$ be the solution of the initial value problem $\dot{x} = Bx$, $x(0) = \eta$. Suppose there is a $C^3$ diffeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$, $h(0) = 0$ such that $h(\phi(t, \xi)) = \psi(t, h(\xi))$ for all $t \in \mathbb{R}$ and all $\xi \in \mathbb{R}^n$. Prove that there is a linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ such that $L(\phi(t, \xi)) = \psi(t, L(\xi))$ for all $t \in \mathbb{R}$, $\xi \in \mathbb{R}^n$.

2. Consider the equation

$$\dot{x} = \frac{t^2 - 1}{t^2 + 1} - \cos x^2$$

(a) Show that if for some $t_0 > 0$ $|x(t_0)| > 100$, then $|x(t)| \to +\infty$ as $t \to +\infty$.

(b) Show that for any $t_0 > 0$ equation (1) has a bounded solution on $[t_0, \infty)$.

3. Consider the equation

$$\ddot{y} + p(t)\dot{y} + q(t)y = 0$$

where $p(t)$ and $q(t)$ are smooth positive functions defined for all $t \in \mathbb{R}$ and $q(t)$ is decreasing.

(a) Rewrite the equation as a linear system and show that the function

$$V(t, y, \dot{y}) = \frac{1}{2}q(t)y^2 + (\dot{y})^2$$

is decreasing along the trajectories of the system.

(b) Recall that a solution $\phi(t)$ is uniformly stable on $[\alpha, \infty)$ if for any $\epsilon > 0$ there is $\delta > 0$ such that for any $t_0 \geq \alpha$ and any solution $\psi(t)$ satisfying $|\psi(t_0) - \phi(t_0)| < \delta$ holds $|\psi(t) - \phi(t)| < \epsilon$ for all $t \geq t_0$. Show that all solutions are uniformly stable on $[0, \infty)$ if $\lim_{t \to \infty} q(t) > 0$.
4 Consider a planar $C^1$ system $\dot{x} = f(x)$. Suppose $f(0) = 0$, and 0 is a nondegenerate critical point, and suppose $C = \{x : |x| = 1\}$ is a periodic orbit. Suppose there are no other fixed points or periodic orbits inside $C$.
Prove that either for all $x : |x| < 1$ $\omega(x)$ coincides with the origin, and $\alpha(x)$ coincides with $C$, or for all $x : |x| < 1$ $\omega(x)$ coincides with $C$, and $\alpha(x)$ coincides with the origin.

5 Consider the boundary value problem

$$u + (\lambda + 32t(1 - t))u = 0, \quad u(0) = u(1) = 0.$$ 
Prove that there is $\epsilon > 0$ such that there are only trivial solutions when $\lambda < \epsilon$.

6 Describe the phase portraits and bifurcations of the parametrized planar system

$$\begin{align*}
\dot{x} &= -y + x(\mu - \sin(x^2 + y^2)) \\
\dot{y} &= x + y(\mu - \sin(x^2 + y^2))
\end{align*}$$