January 2013

Instructions. Answer all six questions. Your work on each question will be assigned a grade from 0 to 10. Your grade will be based on the work that is shown as well as your answer. Carefully show all your steps, justify all your assertions, state precisely any definitions and theorems that you use, and explain your arguments in complete English sentences. Cross out any material that is not to be graded. Some problems may have multiple parts or ask you to do more than one thing. Be sure to go on to subsequent parts even if there is some part you cannot do. If you are asked to prove a result and then apply it to a given situation you may receive partial credit for the correct application.

1. Find an integral for the system

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= x - x^3
\end{align*}
\]  

(1)

and sketch its phase portrait. What are all equilibrium points? Which are Lyapunov stable, Lyapunov asymptotically stable?

2. Consider \( \dot{x} = Ax \), where

\[
A = \begin{bmatrix}
0 & 2 & -2 \\
0 & 1 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]  

(2)

Let \( x(t) \) be a solution of the above system. We are saying that \( x(t) \) is growing linearly if there exists \( \lim_{t \to \infty} \frac{|x(t)|}{t} = c > 0 \), and \( x(t) \) is growing exponentially if \( \lim_{t \to \infty} \ln|x(t)|/t = \chi > 0 \).

Find all initial conditions \( x(0) \) such that the respective solutions \( x(t) \) are

a) bounded;

b) growing linearly;

c) growing exponentially.

In case c) find respective constant \( \chi \).

3. Let \( A \) and \( B \) be real, constant 3 \( \times \) 3 matrices. Suppose there is a map \( h : \mathbb{R}^3 \to \mathbb{R}^3 \) conjugating \( \dot{x} = Ax \) and \( \dot{y} = Ay \). Namely, let \( \phi(t, \xi) \) be the solution of the initial value problem \( \dot{x} = Ax, \ x(0) = \xi \), and let \( \psi(t, \eta) \) be the solution of the initial value problem \( \dot{y} = By, \ y(0) = \eta \).

Then \( h(\phi(t, \xi)) = \psi(t, h(\xi)) \) for all \( t \in \mathbb{R} \) and all \( \xi \in \mathbb{R}^3 \).

a) Suppose eigenvalues of \( A \) and \( B \) are pairwise distinct and \( h \) is a homeomorphism, then the number of eigenvalues with negative and positive real parts of \( A \) and \( B \) is the same.

b) Suppose \( h \) is a \( C^1 \) diffeomorphism, then eigenvalues are the same.

4. Calculate \( \exp(At) \), where

\[
A = \begin{bmatrix}
-1 & 2 & 0 \\
0 & -1 & 2 \\
0 & 0 & -1
\end{bmatrix}
\]  

(3)
5. Consider the system

\[
\begin{aligned}
\dot{x} &= x + \frac{f(x, y)}{x^2 + y^2 + 1} \\
\dot{y} &= y + \frac{g(x, y)}{x^2 + y^2 + 1}.
\end{aligned}
\]  

(4)

Suppose \( f \) and \( g \) are smooth bounded functions. Are there unbounded orbits, critical points? In each case if the answer is yes, give a proof, if the answer is no, give a counterexample.

6. Consider the following system of differential equations on \( \mathbb{R}^2 \):

\[
\begin{aligned}
\dot{x} &= y - x(a - \exp(x^2 + y^2)) \\
\dot{y} &= -x - y(a - \exp(x^2 + y^2)).
\end{aligned}
\]  

(5)

where \( a \) is real constant. Describe the phase portraits and bifurcations of this parametrized planar system.