DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
August, 2015

Probability (Masters Version)

Instructions to the Student

a. Answer all six questions. Each will be graded from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your code number (NOT YOUR NAME) on the outside cover.

c. Keep scratch work on separate pages in the same booklet.

d. If you use a "well known" theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use.

1. There are three coins - two silver and one gold. All the coins are tossed - those that land heads up are kept, those that land heads down are discarded. Then the remaining coins are tossed again, and those that land heads up are kept. This process of tossing the coins and keeping only the ones that land heads up continues until no coins remain. Given that the gold coin was the only one remaining at some point, what is the probability that it took exactly three tosses before there were no coins left.

2. Suppose that $X$ and $Y$ are jointly Gaussian (i.e., $(X,Y)$ is a Gaussian vector). Show that $X$ can be written as $X = aY + Z$, where $a \in \mathbb{R}$, and $Y$ and $Z$ are independent Gaussian variables.

3. Suppose that all the moments of a positive random variable $\xi$ are finite. Prove that $g$ defined via $g(x) = \ln(E\xi^x)$, $x > 0$, is a convex function.
   **Hint:** Recall the inequality $|E(XY)|^2 \leq EX^2EY^2$ that holds for all square-integrable random variables $X$ and $Y$.

4. (a) Give an example of two random variables that are uncorrelated but not independent.
   (b) Give an example of three random variables that are pairwise independent, but not independent.
5. Two players are playing a game where a symmetric coin is tossed repeatedly. If the coin lands heads up three times in a row, the first player wins. If it lands tails up three times in a row, the second player wins.

(a) Prove that the game will end with probability one.

(b) What is the probability that the first player wins the game given that the first time the coin was tossed it landed heads up.

6. Let \( X = \{X_n\}_{n \in \mathbb{N}} \) and \( Y = \{Y_n\}_{n \in \mathbb{N}} \) be two independent sequences of independent Bernoulli random variables, of parameters \( p_1 \) and \( p_2 \), respectively. Let \( T \) be the first success time for \( X \) and \( U \) the first success time for \( Y \) and let \( H = \{T < U\} \).

(a) Compute \( \mathbb{P}(H) \).

(b) Determine the distribution of \( T \) with respect to the conditional probability \( \mathbb{P}_H = \mathbb{P}(\cdot | H) \).