Probability (PhD Version)

Instructions to the Student

a. Answer all six questions. Each will be graded from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your code number (NOT YOUR NAME) on the outside cover.

c. Keep scratch work on separate pages in the same booklet.

d. If you use a "well known" theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use.

1. Let \( \{\xi_n\}_{n \in \mathbb{N}} \) be a sequence of independent and identically distributed random variables taking values in \( \mathbb{Z} \) and let 
\[ \eta_0 := 0, \quad \eta_n := \xi_1 + \cdots + \xi_n, \quad n \in \mathbb{N}. \]
For every \( n \geq 1 \), let \( \theta_n \) denote the number of distinct values taken by the sequence \( \eta_0, \eta_1, \ldots, \eta_n \).

1. Prove that for each \( n \geq 1 \)
\[ \alpha_n := \mathbb{P}(\theta_n = \theta_{n-1} + 1) = \mathbb{P}(\eta_1 \neq 0, \ldots, \eta_n \neq 0). \]

2. Show that
\[ \lim_{n \to \infty} \frac{\mathbb{E}(\theta_n)}{n} = \mathbb{P}(\eta_k \neq 0, \ k \geq 1). \]

2. Let \( \{\xi_n\}_{n \in \mathbb{N}} \) be a Markov chain such that \( \xi_0 \) is uniformly distributed in \([0, 1]\) and
\[
\begin{align*}
\mathbb{P}(\xi_{n+1} = \alpha \xi_n + (1 - \alpha) | \xi_n) &= \xi_n, \\
\mathbb{P}(\xi_{n+1} = \alpha \xi_n | \xi_n) &= 1 - \xi_n,
\end{align*}
\]
for some \( 0 < \alpha < 1 \).
1. Show that \( \{\xi_n\}_{n \in \mathbb{N}} \) is a martingale.

2. Find the almost sure limit of \( \{\xi_n\}_{n \in \mathbb{N}} \) and the limit of \( \{\mathbb{E}(\xi_n)\}_{n \in \mathbb{N}} \).

3. Let \( X \) and \( Y \) be two independent standard Gaussian random variables. For each \( m \in \mathbb{R} \), calculate
\[
\mathbb{E}(XY|Y-2X=m).
\]

4. Let \( \{\xi_n\}_{n \in \mathbb{N}} \) be a sequence of independent random variables. Assume that
\[
\mathbb{P}(\xi_n = 1) = \mathbb{P}(\xi_n = -1) = \frac{1}{2} \left( 1 - \frac{1}{n^2} \right)
\]
and
\[
\mathbb{P}(\xi_n = \sqrt{n}) = \mathbb{P}(\xi_n = -\sqrt{n}) = \frac{1}{2n^2}.
\]
Prove that the Central Limit Theorem holds for \( \xi_1 + \cdots + \xi_n \).

5. Let \( B_t \) be a standard Brownian motion and let
\[
X_t := \exp \left( \lambda B_t - \frac{1}{2} \lambda^2 t \right), \quad t \geq 0.
\]
Show that \( X_t \) is a martingale.

6. Let \( \eta, \xi_1, \xi_2, \ldots \) be independent Poisson random variables of parameter \( \lambda > 0 \) and let
\[
S := \xi_1 + \cdots + \xi_\eta.
\]
Prove that for any function \( \alpha(\lambda) \), such that \( \alpha(\lambda) \to \infty \), as \( \lambda \to +\infty \), we have
\[
\lim_{\lambda \to \infty} \mathbb{P}(S > \lambda^2 \alpha(\lambda)) = 0.
\]