Probability (Ph.D. Version)

Instructions to the Student

a. Answer all six questions. Each will be graded from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your code number (NOT YOUR NAME) on the outside cover.

c. Keep scratch work on separate pages in the same booklet.

d. If you use a "well known" theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use.

1. There are three coins - two silver and one gold. All the coins are tossed - those that land heads up are kept, those that land heads down are discarded. Then the remaining coins are tossed again, and those that land heads up are kept. This process of tossing the coins and keeping only the ones that land heads up continues until no coins remain. Given that the gold coin was the only one remaining at some point, what is the probability that it took exactly three tosses before there were no coins left.

2. Suppose that $X$ and $Y$ are jointly Gaussian (i.e., $(X, Y)$ is a Gaussian vector). Also suppose that $\text{Var}(Y) \neq 0$.
   (a) Show that $X$ can be written as $X = aY + Z$, where $a \in \mathbb{R}$, and $Y$ and $Z$ are independent Gaussian variables.
   (b) Suppose now that $g$ is a polynomial. Prove that the function $f$, defined via $f(y) = E(g(X)|Y = y)$, is also a polynomial.

3. Suppose that all the moments of a positive random variable $\xi$ are finite.
   (a) Prove that $g$ defined via $g(x) = \ln(E\xi^x)$, $x > 0$, is a convex function.
   (b) Prove that $f$ defined via $f(x) = \ln(E\xi^x)/x$, $x > 0$, is a non-decreasing function.

4. (a) Give an example of two random variables that are uncorrelated by not independent.
(b) Give an example of three random variables that are pairwise independent, but not independent.

5. In the following problem, you are allowed to use the results from earlier parts to prove the latter parts, irrespective of whether you solve the earlier parts or not.

(a) Suppose that Gaussian random variables $X_t$ converge to a random variable $X$ almost surely as $t \to 0$. Prove that the convergence also takes place in $L^2$.

(b) Suppose that $X_t$ is a Gaussian random process, i.e., for each $t \in \mathbb{R}$ there is a random variable $X_t = X_t(\omega), \omega \in \Omega$, defined on a probability space $(\Omega, \mathcal{F}, P)$, such that $(X_{t_1}, \ldots, X_{t_n})$ is a Gaussian vector for each $n \geq 1$ and each $t_1, \ldots, t_n \in \mathbb{R}$. Suppose that $X_t$ is a differentiable function for each $\omega$. Prove that $EX_t$ is a differentiable function.

(c) Assume, additionally, that $\text{Var}X_t = \text{const}$. Prove that $X'_t$ is independent of $X_t$ for each fixed $t$.

6. At time $n = 0$, there is one particle on the real line. The particle is located at the origin. At time $n = 1$, the particle splits into two, and each of the resulting particles either jumps one unit to the left or one unit to the right, independently of the other particle. Probabilities of jumping to the left and to the right are both equal to $1/2$. At time $n = 2$, each of the particles again splits into two, and each of the four resulting particles again jumps to the left or to the right with probability $1/2$ each, independently of the other particles and of what happened in the past. This process is continued indefinitely. Note that there are $2^n$ particles immediately after time $n$.

Let $A_n$ be the event that there is at least one particle on the positive semi-axis immediately after time $n$. Prove that $P(A_n) \to 1$ as $n \to \infty$.

**Hint:** you may try to use the fact that even after a rather short time (much smaller than $n$) the number of particles will be very large.