Probability (Ph.D. Version)

Instructions to the Student

a. Answer all six questions. Each will be graded from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your code number (NOT YOUR NAME) on the outside cover.

c. Keep scratch work on separate pages in the same booklet.

d. If you use a "well known" theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use.

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1. Suppose that \( p(n) \) balls are placed randomly and independently into \( n \) boxes. Let \( V_n \) denote the number of empty boxes. Assume that

\[
\lim_{n \to \infty} \frac{p(n)}{n} = \lambda,
\]

for some \( \lambda \geq 0 \).

1. Compute

\[
\lim_{n \to 0} \frac{E(V_n)}{n}.
\]

2. Compute \( \text{Var}(V_n) \).

3. Prove that

\[
\lim_{n \to \infty} \frac{V_n}{n} = e^{-\lambda} \text{ in distribution.}
\]
2. The moment generating function of a random variable \( X \) is defined as
\[
M_X(t) = \mathbb{E}(e^{tX}), \quad t \in \mathbb{R}.
\]
Prove that \( M_X(t) \) is finite in a neighborhood of \( t = 0 \), if and only if there exist \( a, b > 0 \) such that
\[
\mathbb{P}(|X| \geq x) \leq ae^{-bx}, \quad x > 0.
\]

3. Let \( \{X_n\}_{n \geq 1} \) be a Markov chain and let \( \{n_k\}_{k \geq 1} \) be an unbounded increasing sequence of positive numbers. Let us define
\[
Y_k = X_{n_k}, \quad k = 1, 2, \ldots.
\]
1. Show that \( \{Y_k\}_{k \geq 1} \) is a Markov chain (not necessarily time-homogeneous).
2. Determine the transition matrix of \( \{Y_k\}_{k \geq 1} \), when \( n_k = 2k \) and \( \{X_n\}_{n \geq 1} \) is a simple random walk on \( \mathbb{Z} \), i.e.,
\[
\mathbb{P}(X_{n+1} = i + 1 | X_n = i) = 1 - \mathbb{P}(X_{n+1} = i - 1 | X_n = i) = p.
\]

4.
1. Let \( X \) be a random variable such that \( \mathbb{E}|X|^k < +\infty \), for some \( k \geq 1 \). Prove that
\[
\lim_{n \to \infty} n^k \mathbb{P}(|X| > n) = 0 \quad (\ast).
\]
2. Let \( X_1, X_2, \ldots \) be a sequence of independent and identically distributed random variables with finite second moment. By using (\ast), prove that
\[
\lim_{n \to \infty} \frac{1}{\sqrt{n}} \max_{1 \leq j \leq n} |X_j| = 0, \quad \text{in probability.}
\]

5. Let \( X_1, X_2, \ldots \) be a sequence of independent and identically distributed random variables and let \( S_n = X_1 + \cdots + X_n \), for any \( n \geq 1 \). Assume that there exists some \( \delta > 0 \) such that \( \mathbb{E}(e^{\delta X_1}) = 1 \).
1. Prove that \( \{e^{\delta S_n}\}_{n \geq 1} \) is a martingale.
2. Show that for any \( x > 0 \)
\[
\mathbb{P}(S_k \geq x, \text{ for some } k \geq 1) \leq e^{-\delta x}.
\]

6. Let \( \{X_n\}_{n \geq 1} \) be a sequence of independent random variables, uniformly distributed in the interval \([-1, 1]\). Prove that the sequence
\[
\frac{3 \sum_{k=1}^{n} kX_k}{n^{3/2}}, \quad n = 1, 2, \ldots
\]
converges weakly, as \( n \uparrow \infty \). Determine the limiting distribution.

Hint: Remember that \( \sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6. \)