Probability (Ph.D. Version)

Instructions to the Student

a. Answer all six questions. Each will be graded from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your code number (NOT YOUR NAME) on the outside cover.

c. Keep scratch work on separate pages in the same booklet.

d. If you use a “well known” theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use.

1. Suppose that \( \xi_1, \xi_2, \ldots \) is an infinite sequence of independent (but not necessarily identically distributed) random variables. Prove or disprove the following statements:
   (a) If \( \text{Var}(\xi_i) = 1 \) for each \( i \), then the Central Limit Theorem holds.
   (b) If \( 1 \leq \text{Var}(\xi_i) \leq 2 \) and \( \mathbb{E}\xi_i^4 \leq 10 \) for each \( i \), then the Central Limit Theorem holds.

2. There is a box containing two white balls. They added one more ball there (either white or black, with equal probabilities). Then the balls inside the box were mixed, and one was taken out. It turned out to be white. Given this information, what is the probability that the next ball they take out will also be white?

3. Prove that the set of zeros of a Brownian motion (i.e., \( \{ t : W_t = 0 \} \)) is unbounded almost surely.

4. At the end of each month, a king gets a random amount of gold from his subjects. The amount (measured in kilograms) is exponentially distributed with parameter one, and is independent from month to month. What is the average number of months needed to accumulate at least one kilogram of gold? (Assume that he currently has no gold and it’s the beginning of a month.)

5. Suppose that \( \xi_i, i \geq 1 \), is a sequence of random variables that converges in probability. Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is a continuous function. Prove that \( f(\xi_i) \) also converges in probability.

6. Suppose that \( A, B, \) and \( C \) are pairwise independent events such that \( P(A) = P(B) = P(C) \) and \( A \cap B \cap C = \emptyset \). What is the largest possible value for \( P(A) \)?