Probability (Ph.D. Version)

Instructions to the Student

a. Answer all six questions. Each will be graded from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your code number (NOT YOUR NAME) on the outside cover.

c. Keep scratch work on separate pages in the same booklet.

d. If you use a "well known" theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use.

1. Let $\xi$ and $\eta$ be two independent positive random variables on a probability space $(\Omega, \mathcal{F}, P)$ and assume that $\eta$ has exponential distribution with parameter $\lambda > 0$.

   (i) Prove that
   
   $$ P(\eta > \xi) = E(e^{-\lambda \xi}). $$

   (ii) Assume that the event $\mathcal{H} = \{\eta > \xi\}$ has positive probability and let $P_H$ denote the conditional probability given $\mathcal{H}$, that is
   
   $$ P_H(K) = P(K|\mathcal{H}), \quad K \in \mathcal{F}. $$

   Prove that under the probability $P_H$ the random variable $\eta - \xi$ has exponential distribution with parameter $\lambda$ and is independent of $\xi$.

2. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables, with $P(X_i = 1) = P(X_i = -1) = 1/2$. Show that there exist two sequences of constants $\{\alpha_n\}_{n \geq 1}$ and $\{\beta_n\}_{n \geq 1}$ such that

   $$ \lim_{n \to \infty} \frac{(X_1 + 2X_2 + \cdots + nX_n) - \alpha_n}{\beta_n} = \zeta, \quad \text{in distribution}, $$

   where $\zeta \sim N(0, 1)$. 


3. Let $X_0, X_1, X_2, \ldots$ be a sequence of independent Bernoulli trials, with probability of success $p \in (0, 1)$, and let $\nu$ be the first time there are two neighboring successes, that is

$$\nu = \inf \{ n \in \mathbb{N} : X_n = X_{n+1} = 1 \}.$$

Calculate $\mathbb{E} \nu$.

4. Let $X$ and $Y$ be two independent standard Gaussian random variables.

(i) Determine

$$\mathbb{E} (XY \mid X + 2Y).$$

(ii) Compute

$$\mathbb{E} (XY \mid X + 2Y = 3).$$

5. Let $X_1, X_2, \ldots$ be a sequence of independent random variables having standard Gaussian distribution.

(i) Prove that

$$\mathbb{P} \left( X_n > \sqrt{c \log n}, \text{ infinitely often} \right) = 0,$$

for any $c > 2$.

(ii) Prove that

$$\mathbb{P} \left( X_n > \sqrt{2 \log n}, \text{ infinitely often} \right) = 1.$$

(iii) Define

$$Y_n = \frac{X_n}{\sqrt{\log n}}.$$

Prove that the sequence $\{X_n\}_{n \geq 1}$ converges to 0 in probability, but not almost surely.

Hint: Use the inequality

$$\left( x + \frac{1}{x} \right)^{-1} \leq \frac{1 - \Phi(x)}{\varphi(x)} \leq \frac{1}{x}, \quad x > 0,$$

where $\varphi(x)$ and $\Phi(x)$ are respectively the density and the distribution function of the standard normal.

6. Let $\{X_n\}_{n \in \mathbb{N}}$ be a $\{\mathcal{F}_n\}$-martingale, with $X_0 = 0$, $\mathbb{P}$-a.s., and $\mathbb{E}X_n^2 < \infty$, for any $n \in \mathbb{N}$. Prove that

$$\mathbb{E}X_n^2 = \sum_{k=1}^{n} (X_k - X_{k-1})^2, \quad n \in \mathbb{N}.$$