Probability (Ph.D. Version)

Instructions to the Student

a. Answer all six questions. Each will be graded from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your code number (NOT YOUR NAME) on the outside cover.

c. Keep scratch work on separate pages in the same booklet.

d. If you use a “well known” theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use.

1. A die is rolled infinitely many times.
   (a) Find the expected number of combinations (6,6) (i.e., two sixes in a row) among the first 1000 rolls. Find the expected number of combinations (1,2) (i.e., one followed by two) among the first 1000 rolls.
   (b) What is larger and by how much: the expected number of rolls till the first occurrence of (6,6) or the expected number of rolls till the first occurrence of (1,2)?
   (c) What is the probability that (6,6) occurs before (1,2)?

2. We’ll say that a random variable is discrete if the measure it induces is discrete. A random variable is absolutely continuous if the measure it induces is absolutely continuous. Prove or disprove:
   (a) A sum of two discrete random variables is always a discrete random variable.
   (b) A sum of a discrete and an absolutely continuous random variables is always an absolutely continuous random variable.
   (c) A sum of two absolutely continuous random variables is always an absolutely continuous random variable.
   (d) A sum of two independent discrete random variables is always a discrete random variable.
   (e) A sum of a discrete and an absolutely continuous random variables that are independent is always an absolutely continuous random variable.
(f) A sum of two independent absolutely continuous random variables is always an absolutely continuous random variable.

3. Suppose that $\xi_1, \xi_2, ...$ are independent random variables uniformly distributed on the intervals $[-a_1, a_1]$, $[-a_2, a_2]$, ... respectively, where $a_1, a_2, ...$ are positive. Let $\eta_n = \prod_{i=1}^{n} \xi_i$.
   (a) Prove that $\eta_n$ tend to zero almost surely if $\limsup_{n \to \infty} a_n \leq 2$.
   (b) Prove that $\eta_n$ do not have a limit if $\lim_{n \to \infty} (a_n/n^2) = c > 0$.

4. Let $(\mathcal{M}_t, \mathcal{F}_t), t \geq 0$, be a positive continuous martingale that converges to zero almost surely as $t \to \infty$.
   (a) Prove that for each $\lambda > 0$

   $$\mathbb{P}(\sup_{t \geq 0} M_t \geq \lambda | \mathcal{F}_0) = \min(1, M_0/\lambda).$$

   (b) Let $W_t$ be a standard one-dimensional Brownian motion. Prove that for each $a > 0$ the random variable

   $$\sup_{t \geq 0} (W_t - \frac{1}{2} at)$$

   has exponential distribution with parameter $a$. (Hint: apply part (a) to an appropriate martingale.)

5. Suppose that $\xi$ is a bounded random variable on $(X, \mathcal{B}(X), \mathbb{P})$, where $X$ is a metric space, $\mathcal{B}(X)$ is its Borel $\sigma$-algebra, and $\mathbb{P}$ is a probability measure. Suppose that $\xi(x) \neq 0$ for all $x \in X$. Prove that there is a bounded continuous function $\eta$ on $X$ such that $\mathbb{E}(\xi \eta) > 0$.

6. Suppose that $n$ points are independently and uniformly distributed on a unit circle. Let $\xi_n$ be the smallest of the distances between the points. Prove that $n\xi_n$ tends to zero in probability as $n \to \infty$.
   (Hint: One way to approach this is to look at non-intersecting arcs on the circle, whose number tends to infinity as $n \to \infty$, and investigate the probability that one of the arcs contains more than one point.)