Probability (Ph.D. Version)

Instructions to the Student

a. Answer all six questions. Each will be graded from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your code number (NOT YOUR NAME) on the outside cover.

c. Keep scratch work on separate pages in the same booklet.

d. If you use a “well known” theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use.

1. Let $f$ be a continuous bounded function on $\mathbb{R}$. Compute (with justification)

$$
\lim_{n \to \infty} \int_0^\infty \cdots \int_0^\infty f\left(\frac{x_1 + \cdots + x_n}{n}\right) e^{-\left(x_1 + \cdots + x_n\right)} \, dx_1 \cdots dx_n.
$$

2. Let $\xi_1, \xi_2, \ldots$ be independent exponential random variables with parameter 1. Prove that

$$
\limsup_{n \to \infty} \frac{\xi_n}{\ln n} = 1.
$$

3. Suppose that $\xi$ has the following properties:
   (a) $\xi \geq 0$.
   (b) $E\xi = 6$.
   (c) $\text{Var} \xi = 6$.
   By using (a)-(c), give an estimate

$$
P(\xi \geq 10) \leq r
$$

with the constant $r$ that you should choose to be as small as you can.
4. Consider a time-homogeneous Markov chain $X_n$, $n \geq 0$, on the state space $\mathbb{N}$. Suppose that

$$P(X_{n+1} = m - 1 | X_n = m) = P(X_{n+1} = m + 1 | X_n = m) = \frac{1}{2} \text{ if } m > 1, \ m \text{ is odd,}$$

$$P(X_{n+1} = m - 1 | X_n = m) = \frac{2}{3}, \ P(X_{n+1} = m + 1 | X_n = m) = \frac{1}{3} \text{ if } m \text{ is even.}$$

Prove that

$$P(\inf_{n \geq 0} X_n = 1) = 1.$$

5. Let $W_t$ be a Brownian motion. Calculate $E(W_t^2 | W_s)$, $E(W_s|W_t)$, and $E(W_s^2 | W_t)$ for $0 \leq s \leq t$.

Hint: you can write $W_s = \frac{s}{t}W_t + (W_s - \frac{s}{t}W_t)$.

6. Does there exist a Borel-measurable set $A \subseteq [0, 1]$ such that

$$\lambda(A \cap [a, b]) = \frac{1}{2}(b - a)$$

for each interval $[a, b] \subseteq [0, 1]$, where $\lambda$ is the Lebesgue measure on $[0, 1]$. 