Probability (Ph.D. Version)

Instructions to the Student

a. Answer all six questions. Each will be graded from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your code number (NOT YOUR NAME) on the outside cover.

c. Keep scratch work on separate pages in the same booklet.

d. If you use a “well known” theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use.

1. (a) Let $\xi$ be an exponential random variable with parameter $\lambda = 1$. Consider the random variables $\eta_1 = [\xi]$ and $\eta_2 = \{\xi\}$ (the integer and fractional parts of $\xi$). Are $\eta_1$ and $\eta_2$ independent?

(b) Answer the same question for $\xi = |X|$, where $X$ is an $N(0,1)$ (standard normal) random variable.

2. Let $p_n$ be the probability that in a sequence of independent trials with probability of success $p \in (0,1)$ there is an even number of successes in $n$ trials. Prove that

$$p_n = \frac{1}{2} + \frac{1}{2} (1 - 2p)^n.$$

3. Let $\{\xi_n\}$ be a sequence of independent random variables, uniformly distributed on $[0,1]$ and let $S_n = \xi_1 + \cdots + \xi_n$, with $S_0 = 0$.

(a) Prove that for each $n \geq 1$ and $x \in [0,1]$, the density $f_n$ of the random variable $S_n$ is given by

$$f_n(x) = \frac{x^{n-1}}{(n - 1)!}.$$

(b) Let

$$T := \inf \{ n \in \mathbb{N} : S_n > 1 \}.$$
For each \( n \in \mathbb{N} \), compute \( \mathbb{P}(T > n) \).

(c) Compute \( \mathbb{E}(T) \).

4. One million voters will participate in an election in a certain city. The city is divided into 5,000 districts, each with 200 voters. Assume that people vote for a certain candidate (who is very unpopular) with small probability \( p \), independently of each other. Let \( A \) be the event that at least 20,000 people vote in districts where the candidate does not get a single vote. How small should \( p \) be so that \( \mathbb{P}(A) \approx 1/2 \).

5. (a) Prove that a sequence of random variables that is bounded in \( L_p(\Omega, \mathcal{F}, \mathbb{P}) \) for some \( p > 1 \) is uniformly integrable.

   (b) Give an example of a probability space and a sequence of random variables \( \xi_n \) on it, such that \( \xi_n \) is bounded in \( L_2 \), converges in \( L_1 \), but does not converge in \( L_2 \).

6. Let \( W(t) \) be a standard Brownian motion and \( \xi \) a Gaussian random variable with mean \( \mu \) and variance \( \sigma^2 \), independent of \( \{W(t)\}_{t \geq 0} \). We define

\[
X(t) = t\xi + \lambda W(t), \quad t \geq 0,
\]

for some \( \lambda \in \mathbb{R} \), and we set \( \mathcal{F}_t = \sigma(X(s), s \leq t) \).

(a) Compute \( \text{Cov}(\xi, X(s)) \) and \( \text{Cov}(X(t), X(s)) \).

(b) Prove that \( \{X(t)\}_{t \geq 0} \) is a Gaussian process.

(c) Prove that for each \( t \geq 0 \) there exists a constant \( \gamma_t \) such that \( Y(t) := \xi - \gamma_t X(t) \) is independent of \( \mathcal{F}_t \).

(d) Compute \( \mathbb{E}(\xi | \mathcal{F}_t) \) and prove that

\[
\lim_{t \to \infty} \mathbb{E}(\xi | \mathcal{F}_t) = \xi, \quad \mathbb{P} \text{ a.s.}
\]