1. Let \( X_1, \ldots, X_n \) be a sample from a density in the parametric family
\[
f(x, \theta) = \begin{cases} I_{[0,1]}(x) \theta e^{\theta x} / (e^\theta - 1) & \text{for } \theta \neq 0, \\ I_{[0,1]}(x) & \text{for } \theta = 0. \end{cases}
\]
where \( \theta \in \mathbb{R} \) is an unknown parameter.

(a). Find the form of the most powerful test of size \( \alpha = 0.05 \) of the null hypothesis \( H_0 : \theta = 0 \) versus the alternative hypothesis \( H_1 : \theta = 1 \), and explain why it is UMP also for testing \( H'_0 : \theta \leq 0 \) versus \( H_A : \theta > 0 \).

(b). Find the large-sample approximate cutoff defining your rejection region in part (a) as a function of \( n \), and find the approximate power of your test at \( \theta = 1 \) for \( n = 100 \).

2. Let \( Y_1, \ldots, Y_n \) denote a sample from the Poisson(\( \lambda \)) distribution.

(a). Show that \( \sqrt{n} \left\{ \sqrt{\bar{Y}} - \sqrt{\lambda} \right\} \) is an asymptotic pivot, where \( \bar{Y} = n^{-1} \sum_{i=1}^{n} Y_i \), and use it to derive a two-sided confidence interval \([L(\bar{Y}), U(\bar{Y})]\) such that \( P(\lambda > U(\bar{Y})) \approx P(\lambda < L(\bar{Y})) \approx \alpha/2 \).

(b). Find the Likelihood Ratio Test of approximate large-sample size \( \alpha \) based on the sample \( \{Y_i\}_{i=1}^{n} \) of \( H_0 : \lambda = \lambda_0 \) versus \( H_1 : \lambda \neq \lambda_0 \), and use
this family of tests to find the associated two-sided level $1 - \alpha$ confidence interval $[L, U]$ based on $\{Y_i\}_{i=1}^n$.

3. Suppose that $X_1, \ldots, X_n$ is a Uniform[0, 1] data sample, for large $n$.
   (a) Find the large-sample limiting joint distribution of $\sqrt{n} (\bar{X} - 1/2)$ and $\sqrt{n}(S^2 - 1/12)$, where $\bar{X} = n^{-1} \sum_{i=1}^n X_i$ and $S^2 = (n-1)^{-1} \sum_{i=1}^n (X_i - \bar{X})^2$.
   (b) Find the large-sample limiting distribution of $\sqrt{n} (\bar{X}/S - \sqrt{3})$.

4. Find the relative efficiency of the Method of Moments Estimator for $\lambda$ based on a sample $Z_1, \ldots, Z_n$ from the parametric density
   \[ f(z, \lambda) = \frac{2\sqrt{\lambda}}{\sqrt{\pi}} e^{-\lambda z^2}, \quad z > 0 \]

5. Suppose that $Y_1, \ldots, Y_n$ are iid observations from the density
   \[ f(y, \theta, \beta) = \beta e^{-\beta (y-\theta)} I_{[y \geq \theta]} \]
   where $\beta > 0, \theta \in \mathbb{R}$ are unknown parameters.
   (a). Show that $(T_1, T_2) = (\min(Y_1, \ldots, Y_n), \bar{Y})$ are sufficient for the parameter $(\theta, \beta)$, that $T_1$ is complete sufficient for $\theta$ when $\beta$ is fixed and known, and that $T_2$ is complete sufficient for $\beta$ when $\theta$ is fixed and known.
   (b). Show that $T_2 - T_1$ is independent of $T_1$ for all values of $(\theta, \beta)$.

6. Suppose that iid data-vectors $V_i = (X_i, Y_i)$ for $i = 1, \ldots, n$ are observed with $X_i \sim \text{Poisson}(\lambda), Y_i \sim \text{Poisson}(1/\lambda)$ independent for each $i$, where $\lambda > 0$ is an unknown parameter.
   (a) Show that the joint probability mass function of the sample $\{V_i\}_{i=1}^n$ is a natural exponential family, and give its natural parameter $\theta$ as a function of $\lambda$, and its natural parameter space.
   (b) Give the form of a conjugate family of prior densities for the unknown parameter $\lambda$. 