Unless otherwise stated, you may appeal to a "well known theorem" in your solution to a problem, but if you do, it is your responsibility to make it clear which theorem you are using and why its use is justified. In problems with multiple parts, be sure to go on to the rest of the problem even if there is some part you cannot do. In working on any part, you may assume the answer to any previous part, even if you have not proved it.

Problem 1.

As usual, let \( \mathbb{R}^n \) have its usual topology and let \( S^{n-1} \) be the unit sphere in \( \mathbb{R}^n \).

a) Show that \( S^n \) is connected for \( n > 0 \) and compact for all \( n \).

b) Embed \( \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1} \) via \( (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0) \) and let \( \mathbb{R}^\infty = \lim_{n \to \infty} \mathbb{R}^n \) be the inductive limit (infinite union) of the \( \mathbb{R}^n \) as \( n \to \infty \), with the \textit{inductive limit topology} in which a subset \( C \) is closed if and only if \( C \cap \mathbb{R}^n \) is closed for each \( n \). Let \( S^{\infty} = \lim_{n \to \infty} S^{n-1} \) have the subspace topology from \( \mathbb{R}^\infty \). Show that \( S^{\infty} \) is connected but not compact.

Problem 2.

Let \( D \) be the closed unit disc in \( \mathbb{R}^2 \) and \( S^1 = \partial D \) the boundary of \( D \). Let \( f, g : S^1 \to \mathbb{R}^3 \) be smooth embeddings such that \( f(S^1) \cap g(S^1) = \emptyset \). Define \( \lambda : S^1 \times S^1 \to S^2 \) by

\[
\lambda(x, y) = \frac{f(x) - g(y)}{||f(x) - g(y)||}.
\]

a) Show \( \lambda \) is a smooth map.

b) Suppose \( f \) extends to a smooth map \( \hat{f} : D \to \mathbb{R}^3 \) with \( \hat{f}(D) \cap g(S^1) = \emptyset \). Show that the degree of \( \lambda \) is 0.

c) Let \( (x_1, x_2) \) be coordinates on \( S^1 \subset \mathbb{R}^2 \) and suppose \( f(x_1, x_2) = (x_1, x_2, 0) \) and \( g(x_1, x_2) = (0, x_1 + 1, x_2) \). Show that the degree of \( \lambda \) is \textit{non-zero}.

Problem 3.

Let \( X \) denote the CW-complex obtained from 3-simplices by identifying faces via the unique simplicial identifications preserving the arrows (see Figure 1).

a) Show that \( X \) has one 0-cell, three 1-cells, six 2-cells, and three 3-cells.

b) Write down the cellular chain complex for \( X \).
c) The homology groups of $X$ are given by

$$H_n(X) = \begin{cases} 
\mathbb{Z} & \text{for } n = 0, 2, 3 \\
\mathbb{Z}/3\mathbb{Z} & \text{for } n = 1 \\
0 & \text{otherwise.}
\end{cases}$$

Verify this for $n = 1$ and $n = 3$.

d) Does $X$ have the homotopy type of a closed 3-manifold? Why or why not? You may make use of the homology groups in (c), even the ones you didn’t compute.

**Problem 4.**

Let $K^2$ be the Klein bottle.

a) Exhibit $K^2$ as the boundary of a compact 3-manifold.

b) Can $K^2$ be the boundary of an orientable 3-manifold? Why or why not?

**Problem 5.**

Let $X \subseteq \mathbb{R}^3$ be the subset obtained by rotating the union of two tangent circles in the plane around a disjoint axis in the same plane parallel to a line joining the centers of the two circles. (See Figure 2.) Compute $\pi_1(X)$ and the integral homology groups of $X$. Describe a non-trivial covering space of $X$.

![Figure 2. Rotating two tangent circles.](image)

**Problem 6.**

a) Let $m, n \geq 1$. Describe the cohomology rings $H^*(\mathbb{R}P^m \vee \mathbb{R}P^n, \mathbb{Z}_2)$ and $H^*(\mathbb{R}P^n \times \mathbb{R}P^m, \mathbb{Z}_2)$.

b) Show that $\mathbb{R}P^m \vee \mathbb{R}P^n$ cannot be a retract of $\mathbb{R}P^n \times \mathbb{R}P^m$ for $n, m \geq 1$. 