Unless otherwise stated, you may appeal to a "well known theorem" in your solution to a problem, but if you do, it is your responsibility to make it clear which theorem you are using and why its use is justified. In problems with multiple parts, be sure to go on to the rest of the problem even if there is some part you cannot do. In working on any part, you may assume the answer to any previous part, even if you have not proved it.

**Problem 1.**

Let $p: X' \to X$ and $q: Y' \to Y$ be covering maps, with all spaces Hausdorff, and suppose $f: X \to Y$ is a continuous map. Let $f_1, f_2: X' \to Y'$ be continuous maps such that $q \circ f_i = f \circ p$ for $i = 1, 2$. Let $E = \{x' \in X' \mid f_1(x') = f_2(x')\}$. 

1. Show $E$ is open and closed in $X'$.
2. Suppose $X'$ is connected and $q$ is regular (i.e., Galois). Then there exists a deck (or covering) transformation $\tau: Y' \to Y'$ of $q: Y' \to Y$ such that $f_2 = \tau \circ f_1$.

**Problem 2.**

Let $Y = S^1 \vee S^1$ the wedge of two circles with fundamental group $F(a, b)$ the free group on two generators. Let $X$ be the CW-complex obtained from $Y$ by attaching two 2-cells, one by the map $\alpha: S^1 \to Y$ whose homotopy class is $aba^{-1}b^{-1}$ and one by $\beta: S^1 \to Y$ whose homotopy class is $a^2$.

1. Determine $\pi_1(X)$, the Euler characteristic of $X$, and the integral homology of $X$.
2. Prove or disprove: $X$ has the homotopy type of a surface.

**Problem 3.**

1. Let $M$ be a compact, connected smooth manifold of dimension $n > 1$. Let $f: M \to S^n$ be a smooth map such that $df_x: TM_x \to TS^n_{f(x)}$ has rank $n$ for all $x \in M$. Show that $f$ is a diffeomorphism $M \to S^n$. Show the statement is false if $n = 1$.
2. Let $\alpha \in \mathbb{R}$ and

   $$M = \{(z_0, z_1, z_2, z_3) \in \mathbb{RP}^3 \mid (z_0 - z_3)^2 + \alpha z_1^2 = 0\}.$$ 

   Show $M$ is a smooth submanifold of $\mathbb{RP}^3$ of dimension 2 when $\alpha = 0$, but not if $\alpha \neq 0$.

**Problem 4.**

Let $M^n$ be an $n$-dimensional oriented, compact manifold and let $S^p \times D^{n-p} \subseteq M$ where $n > p > 0$. ($S^q$ is the $q$-dimensional sphere and $D^k$ is the $k$-dimensional closed disc.) Consider the manifold

$$M' = \left( M - \{ S^p \times D^{n-p} \} \right) \cup \bigcup_{S^p \times S^{n-p-1}} (D^{p+1} \times S^{n-p-1})^1.$$
That is, remove \( S^p \times D^{n-p} \) from \( M \) leaving a manifold \( \overline{M} \) with boundary \( S^p \times S^{n-p-1} \) which is also the boundary of \( D^{p+1} \times S^{n-p-1} \). Then \( M' = \overline{M} \cup (D^{p+1} \times S^{n-p-1}) \) with the boundaries identified. This is called a \( (p, n-p) \) surgery on \( M \).

1. Compute the difference between the Euler characteristics of \( M \) and \( M' \).
2. Use your answer to (1) to show that you cannot convert \( S^2 \times S^2 \) to \( \mathbb{CP}^2 \) through any sequence of surgeries.

**Problem 5.**

Consider the \((2n+1)\)-sphere \( S^{2n+1} \) as \( \{ (z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} : |z_0|^2 + \cdots + |z_n|^2 = 1 \} \). \( S^1 \) acts on \( S^{2n+1} \) by multiplication on each coordinate. The quotient space is by definition \( \mathbb{CP}^n \). Observe that \( \mathbb{CP}^1 \) is naturally homeomorphic to \( S^2 \). Therefore we may consider the quotient map \( S^3 \to \mathbb{CP}^1 \) as a map \( f: S^3 \to S^2 \).

1. Show the complex projective \( \mathbb{CP}^2 \) is homeomorphic to the mapping cone
\[
C_f = S^2 \cup_f D^4 = (S^2 \coprod D^4)/\{ x \sim f(x) : x \in S^3 \}.
\]
2. Use cohomology to show that although \( 0 = f_*: H_k(S^3) \to H_k(S^2) \) for \( k > 0 \), \( f \) is not homotopic to a constant map.

**Problem 6.**

Let \( M^n \) be a closed orientable manifold of dimension \( n \).

1. If \( n \) is odd show \( \chi(M) = 0 \).
2. If \( n \equiv 2 \mod 4 \) show \( \chi(M) \) is even. (It may be useful to know that if \( A \) is a skew-symmetric, non-singular \( n \times n \) real matrix, then \( n \) is even.)
3. If \( n = 4k \) and \( s \) is any integer, show there exists a closed orientable manifold \( M^n \) with \( \chi(M) = s \). (Hint: use Problem 4.)